Solutions for Take-Home Final Exam.

1. Solve the Dirichlet problems

(a) \[ M^+ u := \sup_{L} \{ Lu + 1 \} = 0 \text{ in } B_1, \quad u = 0 \text{ on } \partial B_1; \]

(b) \[ M^+ u := \inf_{L} \{ Lu + 1 \} = 0 \text{ in } B_1, \quad u = 0 \text{ on } \partial B_1, \]

where \( B_1 := \{ x \in \mathbb{R}^n : |x| < 1 \} \), and operators \( L \) have the form

\[ Lu := \sum_{i,j} a_{ij} D_{ij} u + \sum_i b_i D_i u, \]

with \( a_{ij} \) and \( b_i \) satisfying the conditions

\[ a_{ij} = a_{ji}, \quad \nu |\xi|^2 \leq \sum_{i,j} a_{ij} \xi_i \xi_j \leq \nu^{-1} |\xi|^2 \quad \text{for all} \quad \xi \in \mathbb{R}^n; \quad |b| := \left( \sum_i b_i^2 \right)^{1/2} \leq K, \]

with constants \( \nu \in (0, 1] \) and \( K \geq 0 \).

**Solution.** Since the problem is radially symmetric, one can expect that the solution depends only on \( r := |x| \), i.e. \( u(x) = v(r) \). In the probabilistic interpretation, the problem (a) corresponds to maximization of exit time out of \( B_1 \) for a diffusion process starting from the point \( x \in B_1 \), so that the diffusion should be minimal possible, and the drift \( b \) should be directed to the origin \( 0 \in B_1 \). Then the equation in (a) takes the form

\[ \nu \left( v'' + \frac{n-1}{r} \cdot v' \right) - K v' + 1 = 0 \quad \text{on} \quad (0, 1], \quad v'(0) = v(1) = 0. \]

Here we used the fact that by the maximum principle, \( v' < 0 \), and also \( v'' < 0 \), which will be established soon. The function \( w := r^{n-1} v' \) satisfies

\[ \nu w' - K w + r^{n-1} = 0, \quad w(0) = 0. \]

The solution to this problem is

\[ w(r) = -\nu^{-1} \int_0^r t^{n-1} e^{K(r-t)/\nu} \, dt = -\nu^{-1} r \int_0^1 s^{n-1} e^{K(r-s)/\nu} \, ds. \]

Then

\[ v'(r) = r^{1-n} w(r) = -\nu^{-1} r \int_0^1 s^{n-1} e^{K(r-s)/\nu} \, ds. \]

From this expression we get \( v''(r) < 0 \) for \( r > 0 \). Finally,

\[ v(r) = -\int_0^1 v'(t) \, dt = \nu^{-1} \int_0^1 \int_0^1 t s^{n-1} e^{K(t-s)/\nu} \, ds \, dt. \]

The solution to the problem (b) is quite similar, one only needs to replace \( \nu \) by \( \nu^{-1} \) and \( K \) by \( -K \).
It is possible to write down the above integral as a finite sum of elementary functions, but this will not make it more elegant. We only point out that in the case \( n = 1 \), the answer is reduced to

\[
u(x) = \frac{\nu}{K^2} \left( e^{K/x} - e^{K|x|/\nu} \right) + \frac{|x| - 1}{K} \quad \text{for} \quad -1 \leq x \leq 1.
\]

Despite the presence of \(|x|\) in this expression, the function \( u(x) \) is of class \( C^2 \), as one can see from the expansion

\[
e^{K|x|/\nu} = 1 + \frac{K|x|}{\nu} + \frac{K^2 x^2}{2\nu^2} + \cdots.
\]

2. Let \( u(t, x) \) be the bounded classical solution to the Cauchy problem

\[
u_t = a(x) u_{xx} \quad \text{for} \quad (t, x) \in (0, \infty) \times \mathbb{R}, \quad \text{where} \quad a(x) := 1 - \varepsilon \sin x, \quad 0 < \varepsilon < 1,
\]

with the initial data \( u(0, x) = u_0(x) := \sin x \). Show that there exists a constant \( c > 0 \), such that

\[
\sup_{x \in \mathbb{R}} |u(t, x) - c| \to 0 \quad \text{as} \quad t \to +\infty.
\]

**Proof.** From the uniqueness of the bounded solution it follows that \( u(t, x) \) is periodic with respect to \( x \), i.e. \( u(t, x + 2\pi) = u(t, x) \). The problem consists of two parts: (i) the property (2) with a constant \( c \), and (ii) the claim that \( c > 0 \). By the maximum principle,

\[
M(t) := \sup_{x} u(t, x) \quad \text{is non-increasing}, \quad \text{and} \quad m(t) := \inf_{x} u(t, x) \quad \text{is non-decreasing on} \quad [0, \infty).
\]

(i) The first part can be re-formulated as \( M(t) - m(t) \to 0 \) as \( t \to +\infty \). This fact is of very general nature, it follows easily from the Harnack inequality or Hölder estimates for solutions, even for equations with measurable coefficients. It is also possible to derive it by more elementary means as follows. Introduce the functions

\[
I_1(t) := \int_{-\pi}^{\pi} u_x^2(t, x) \, dx, \quad I_2(t) := \int_{-\pi}^{\pi} u_{xx}^2(t, x) \, dx.
\]

Using integration by parts, we derive

\[
\frac{dI_1}{dt} = 2 \int_{-\pi}^{\pi} u_x u_{xt} \, dx = -2 \int_{-\pi}^{\pi} u_{xx} u_t \, dx = -2 \int_{-\pi}^{\pi} u_{xx}^2 \, dx \leq -c_1 I_2, \quad \text{where} \quad c_1 := 2(1 - \varepsilon) > 0.
\]

Further, for every \( t \geq 0 \), there exist points \( x_0, x_1 \in (-\pi, \pi] \) such that \( m(t) = u(t, x_0) \) and \( M(t) = u(t, x_1) \), so that

\[
0 \leq M(t) - m(t) = \int_{x_0}^{x_1} u_x(t, x) \, dx \leq \int_{-\pi}^{\pi} |u_x(t, x)| \, dx \leq (2\pi \cdot I_1(t))^{1/2}.
\]

We also have \( u_x(t, x_0) = u_x(t, x_1) = 0 \), therefore, by integration,

\[
\sup_{x} |u_x| \leq \pi |u_{xx}| \, dx, \quad \text{and} \quad I_1 \leq 2\pi \cdot \sup_{x} u_x^2 \leq 2\pi \cdot \left( \int_{-\pi}^{\pi} |u_{xx}| \, dx \right)^2 \leq 4\pi^2 \cdot I_2.
\]
The last estimate together with (3) imply
\[ \frac{dI_1}{dt} \leq -cI_1, \quad \text{where} \quad c := \frac{c_1}{4\pi^2} > 0, \quad \text{hence} \quad I_1(t) \leq e^{-ct}I_1(0) \to 0 \quad \text{as} \quad t \to +\infty. \]

Now the desired property \( M(t) - m(t) \to 0 \) as \( t \to +\infty \) follows immediately from (4).

(ii). In order to prove that \( c > 0 \), note that
\[ I(t) := \int_0^{\pi} \frac{u(t, x) \, dx}{a(x)} = I(0), \quad \text{because by periodicity,} \quad \frac{dI}{dt} = \int_0^{\pi} \frac{u_t \, dx}{a} = \int_0^{\pi} u_{xx} \, dx = 0. \]

Therefore, using the convergence \( u \to c \) in the previous part (i), we derive
\[ I(0) = \lim_{t \to +\infty} I(t) = cA, \quad \text{where} \quad A := \int_0^{\pi} \frac{dx}{a(x)}. \]

Since \( c = I(0)/A \), it remains to show that both \( I(0) \) and \( A \) are strictly positive. Indeed, this is the case:
\[ A := \int_0^{\pi} \frac{dx}{1 - \varepsilon \sin x} = \int_0^{\pi} \left( \frac{1}{1 - \varepsilon \sin x} + \frac{1}{1 + \varepsilon \sin x} \right) \, dx = \int_0^{\pi} \frac{2 \, dx}{1 - \varepsilon^2 \sin^2 x} > 0, \]
\[ I(0) = \int_0^{\pi} \frac{\sin x \, dx}{1 - \varepsilon \sin x} = \int_0^{\pi} \sin x \cdot \left( \frac{1}{1 - \varepsilon \sin x} - \frac{1}{1 + \varepsilon \sin x} \right) \, dx = \int_0^{\pi} \frac{2 \varepsilon \sin^2 x \, dx}{1 - \varepsilon^2 \sin^2 x} > 0. \]

\[ \square \]

3. Let \( f(u) \) be a non-negative, non-decreasing, continuous function on \( \mathbb{R}^1 \). Show that the problem
\[ \begin{cases} \Delta u = f(u) & \text{in} \quad B_R := \{ x \in \mathbb{R}^n, \, |x| < R \}, \\ u(x) \to +\infty & \text{as} \quad d(x) := \text{dist}(x, \partial B_R) = R - |x| \to 0, \end{cases} \tag{5} \]
cannot have more than one solution in \( C^2_{\text{loc}}(B_R) \).

**Proof.** Step I. Let \( u_1 \) and \( u_2 \) be two different solutions to the problem (5). Interchanging \( u_1 \) and \( u_2 \) if necessary, one can assume that \( u_2 - u_1 > 0 \) at some point \( x_0 \in B_R \). Introduce two functions
\[ u(r) := \min_{|x|=r} u_1(x) \quad \text{and} \quad \upsilon(r) := \max_{|x|=r} u_2(x) \quad \text{for} \quad 0 \leq r < R. \tag{6} \]

We claim that the difference \( \omega(r) := (\upsilon - u)(r) \) is positive and strictly increasing:
\[ 0 < \omega(a) < \omega(b) \quad \text{for} \quad r_0 := |x_0| < a < b < R. \tag{7} \]

Note that \( \omega(r_0) \geq (u_2 - u_1)(x_0) > 0 \). Fix two points \( a < b \) in \((r_0, R)\). Since the problem (5) is invariant with respect to rotations in \( \mathbb{R}^n \), one can replace \( u_2(x) \) by \( u_2(Tx) \) for any rotation \( T \). Therefore, without loss of generality, we can assume that
\[ u(a) = u_1(x_1) \quad \text{and} \quad \upsilon(a) = u_2(x_1) \quad \text{at the same point} \quad x_1 \in \partial B_a. \]

Under such rotations, \( \omega(r_0) > 0 \) remains the same, therefore, the set \( \Omega := B_b \cap \{u_2 > u_1\} \) is open and non-empty. Moreover, since \( f(u) \) is monotone, we also have
\[ \Delta(u_2 - u_1) = f(u_2) - f(u_1) \geq 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad u_2 - u_1 = 0 \quad \text{on} \quad (\partial\Omega) \cap B_b. \]
We must have $\omega(a) > 0$, because otherwise $u_2 - u_1 \leq 0$ on $\partial B_a$, and $u_2 - u_1 = 0$ on the boundary of the non-empty set $\Omega \cap B_a$, which is impossible by the maximum principle. Now by the strong maximum principle (Theorem 3.5 in [1]), the claim (7) follows:

$$0 < \omega(a) = (u_2 - u_1)(x_1) < \sup_{\partial B_a} (u_2 - u_1) = \sup_{\partial B_b} (u_2 - u_1) \leq (\pi - u)(b) = \omega(b).$$

**Step II.** Ideally, if functions $u$ and $\overline{u}$ were of class $C^2$, we could proceed as follows. By our construction in (6), we have

$$u(|x|) \leq u_1(x), \quad \overline{u}(|x|) \geq u_2(x) \quad \text{for every } x \in B_R,$$

with equalities attained at some points. Since both functions $u_1$ and $u_2$ satisfy $\Delta u = f(u)$, at the points of “contact”,

$$\Delta u(|x|) \leq \Delta u_1(x) = f(u_1(x)) = f(u(|x|)), \quad \text{and similarly, } \Delta \overline{u}(|x|) \geq f(\overline{u}(|x|)).$$

In general, even for smooth functions $u(x)$, the corresponding functions $u(r)$ and $\overline{u}(r)$ in (6) may not belong to $C^1$, whereas the previous requires the smoothness of class $C^2$. The idea is that if we do have $C^2$-smoothness, then the functions $v_1(r) := u(r)$ and $v_2(r) := \overline{u}(r)$, where $r := |x|$, satisfy

$$\Delta v_1 := v_1'' + \frac{n-1}{r} \cdot v_1 \leq f(v_1), \quad \Delta v_2 \geq f(v_2) \quad \text{for } 0 < r < R, \quad \text{and} \quad v_{1,2}(r) \to +\infty \quad \text{as} \quad r \searrow R,$$

and from strict monotonicity of $\omega(r) = v_2(r) - v_1(r)$ it follows that there are points $a < R$, which can be chosen arbitrarily close to $R$, at which

$$v_1(a) < v_2(a), \quad v_1'(a) < v_2'(a). \quad (9)$$

Then the consideration of the ODE (8) in the interval $(a, R)$ with the initial conditions (9) provides the desired contradiction (see Step V below).

**Step III.** In general, even for smooth functions $u(x)$, the corresponding functions $u(r)$ and $\overline{u}(r)$ in (6) may not belong to $C^1$, whereas the previous Step II requires the smoothness of class $C^2$. We will bypass this step as follows.

Fix two points $a$ and $b$ satisfying (7). Since $\omega(a) < \omega(b)$, one can choose the constants $A \in (u(a), \overline{u}(a))$ and $B \in (u(b), \overline{u}(b))$, such that

$$0 < A - u(a) < B - u(b) \quad \text{and} \quad 0 < \overline{u}(a) - A < \overline{u}(b) - B. \quad (10)$$

Now consider the classical (in $C^2$) solution to the problem

$$\Delta v = f(v) \quad \text{on } [a, b], \quad v(a) = A, \quad v(b) = B. \quad (11)$$

For example, one can prove the existence of such a solution by the method of “shooting”: consider the initial value problems

$$\Delta v(r) = f(v(r)) \quad \text{for } r \geq a, \quad v(a) = A, \quad v'(a) = C, \quad (12)$$

with a parameter $C$. One can show that the values of $v$ at every point depend monotonically and continuously on $C$, so that the value $v(b) = B$ can be attained under the appropriate choice of $C$. It is not difficult to derive there facts from the observation that the equation $\Delta v = f(v)$ can be re-written as

$$r^{n-1} v'(r) = f(v),$$

and then from (12) it follows

$$r^{n-1} v'(r) = a^{n-1} C + \int_a^r s^{n-1} f(v(s)) \, ds \quad \text{for } r \geq a. \quad (13)$$
For more details and further implications, see [4], Sec. 2, with $\alpha = n - 1$ and $p = 2$.

**Step IV.** We claim that under the conditions (10), the function $v$ in (11), (12) can be extended as the solution of $\Delta v = f(v)$ in class $C^2_{loc}$ to the interval $[a, R]$, and

$$u(r) < v(r) < \overline{u}(r) \quad \text{for} \quad b \leq r < R. \tag{14}$$

Indeed, if these inequalities fail, then the graph of $v(r)$ must intersect at least one of graphs of $u(r)$ or $\overline{u}(r)$ for $b \leq r < R$, i.e.

$$\tau_1 := \min\{r \geq b: v(r) = u(r)\} < R \quad \text{or} \quad \tau_2 := \min\{r \geq b: v(r) = \overline{u}(r)\} < R.$$

Suppose that $\tau_1 < R$. Then the function $w_1(x) := v(|x|) - u_1(x)$ satisfies

$$\Delta w_1 = f(v) - f(u_1) \geq 0 \quad \text{on the set} \quad \Omega_1 := \{x \in \mathbb{R}^n: a < |x| < \tau_1, \ w_1(x) > 0\}.$$

Note that the maximum of $w_1$ on $\partial\Omega_1$ is attained on $\partial B_a$, because $w_1 \leq 0$ on $\partial\Omega_1 \setminus (\partial B_a)$. Using (6), (10), (11), and the maximal principle applied to $w_1$ on $\Omega_1$, we obtain the desired contradiction:

$$0 < B - u\overline{y}(b) = \sup_{\partial B_a} w_1 \leq \sup_{\partial\Omega_1} w_1 = \sup_{\partial\Omega_1} w_1 = A - \overline{u}(a) = \overline{u}(b) < B - \overline{u}(b).$$

Now suppose that $\tau_2 < R$. Similarly, consider the function

$$w_2(x) := u_2(x) - v(|x|) \quad \text{on the set} \quad \Omega_2 := \{x \in \mathbb{R}^n: a < |x| < \tau_2, \ w_2(x) > 0\}.$$

In this case, we also get a contradiction as follows:

$$0 < B - \overline{u}(b) = \sup_{\partial B_a} w_2 \leq \sup_{\partial\Omega_2} w_2 = \sup_{\partial\Omega_2} w_2 = A - \overline{u}(a) = \overline{u}(b).$$

**Step V.** It remains to show that the relations (14) are impossible. Now we use again the continuous dependence of the parameter $B$ in (11) on $A$ and $C$ in (12). By slightly modifying these parameters, we can get two solutions $v_1$ and $v_2$ of $\Delta v = f(v)$ on the interval $[a, R]$, which blow up as $r \nearrow R$, with different initial data satisfying (9). From (13) it follows that the inequalities $v_1 < v_2$ and $v'_1 < v'_2$ remain valid on the whole interval $[a, R]$, and moreover, replacing $a$ by a larger value if necessary, we can assume that both $v_1$ and $v_2$ satisfy

$$0 < v' \leq C + \frac{r}{n} \cdot f(v), \quad v'' = f(v) - \frac{n - 1}{r} \cdot v' \geq \frac{1}{n} \cdot f(v) - \frac{(n - 1)C}{a} \quad \text{for} \quad a \leq r < R. \tag{15}$$

Since the right hand side converges to $+\infty$ as $r \nearrow R$, one can further increase $a$ to guarantee that

$$a \in [3R/4, R], \quad v_2 > v_1 > 0, \quad v'_2 > v'_1 > 0, \quad \text{and} \quad v''_2 > 0 \quad \text{on} \quad [a, R].$$

Consider the following “perturbation” of $v_1(r)$:

$$v_0(r) := v_1(r_{\varepsilon}), \quad \text{where} \quad r_{\varepsilon} := (1 - \varepsilon)r + \frac{3\varepsilon R}{2} \leq (1 + \varepsilon)r < \frac{r}{1 - \varepsilon} \quad \text{for} \quad r \geq a \geq \frac{3R}{4}, \tag{16}$$

and $\varepsilon > 0$ is chosen so small that

$$v_2(a) > v_0(a), \quad v'_2(a) > v'_0(a). \tag{17}$$
Note that
\[ v_0(r) \to +\infty \quad \text{as} \quad r \nearrow R_0 := \frac{(2 - 3\varepsilon)R}{2(1 - \varepsilon)} < R. \quad (18) \]
Moreover, since \( v''_1 > 0, v'_1 > 0 \), from (16) it follows
\[
\Delta v_0(r) := v''_0(r) + \frac{n - 1}{r} \cdot v'_0(r) = (1 - \varepsilon)^2 v''_1(r_\varepsilon) + \frac{(1 - \varepsilon)(n - 1)}{r} \cdot v'_1(r_\varepsilon) < v''_1(r_\varepsilon) + \frac{n - 1}{r_\varepsilon} \cdot v'_1(r_\varepsilon) = \Delta v_1(r_\varepsilon) = f(v_1(r_\varepsilon)) = f(v_0(r)) \quad \text{on} \quad [a, R_0).
\]
By virtue of (17) and (18), the function \( w_0 := v_2 - v_0 \) has a local maximum at some point \( a_0 \in (a, R_0) \).
At this point, combining together the previous relations, we get the desired contradiction:
\[ w_0 > 0, \quad w'_0 = 0, \quad w''_0 \leq 0, \quad \text{and} \quad 0 \geq \Delta w_0 = \Delta v_2 - \Delta v_0 > f(v_2) - f(v_0) \geq 0. \]
The proof is complete. \( \square \)

**Remark.** It is well known that there exists a “blow-up” solution to the problem (5) if and only if the non-decreasing function \( f \) satisfies the following *Keller-Osserman condition* (see [2], [3])
\[
\int_A^\infty \frac{dt}{\sqrt{tf(t)}} < \infty. \quad (K-O)
\]
The uniqueness of a solution is also known, and it is essentially contained in [4], Theorem 6. It was shown there (for more general equations), using transformation (16), that solutions to the problems
\[
\Delta v := v'' + \frac{n - 1}{r} \cdot v' = f(v) \quad \text{for} \quad 0 < r < R, \\
v(0) = A, \quad v'(0) = 0, \quad \text{and} \quad v(r) \to +\infty \quad \text{as} \quad r \nearrow R,
\]
continuously depend on parameters \( A \) and \( R \), with strictly monotone dependence between these two parameters. Then the function \( u(x) := v(|x|) \) is a solution to the problem (5) in the class \( C^{2}_{\text{loc}} \). If \( u_R(x) \) denotes such a radially symmetric solution in \( B_R \), then by elementary comparison argument any (a priori possibly non-radially symmetric) solution \( u \) is “squeezed” between radially symmetric solutions:
\[
u_{R-\varepsilon} > u > u_{R+\varepsilon} \quad \text{in} \quad B_{R-\varepsilon} \quad \text{for} \quad \varepsilon > 0.
\]
Then the uniqueness follows from the continuous dependence:
\[
u = u_R = \lim_{\varepsilon \to 0^+} u_{R\pm\varepsilon}.
\]

**References.**