Appendix A. Ordered sets

This note is supplementary to the book:

In our exposition, we partially follow Chapter I, §6 in the book:

Here we derive the Well Ordering Principle and the Hausdorff Maximal Principle from

**The Axiom of Choice.** For an arbitrary non-empty set \( X \), there exists a function

\[
    f : 2^X \setminus \emptyset \mapsto X \quad \text{such that} \quad f(A) \in A \quad \text{for every non-empty set} \quad A \subseteq X. \tag{1}
\]

Throughout this note, we assume that \( X \) is a non-empty set.

A set \( X \) is **partially ordered** by a relation “\( x \leq y \)” for some pairs \( (x, y) \in X \times X \) if

(i) \( x \leq y, \ y \leq z \implies x \leq z \);
(ii) \( x \leq y, \ y \leq x \implies x = y \);
(iii) \( x \leq x \) for every \( x \in X \).

We write \( x < y \) iff \( x \leq y, \ x \neq y \).

A partially ordered set \( (X, \leq) \) is **linearly (totally) ordered**, or \( (X, \leq) \) is a **chain**, if for every pair \( (x, y) \in X \times X \) we have either \( x \leq y \) or \( y \leq x \).

A linearly ordered set \( (X, \leq) \) is **well ordered** if every non-empty set \( A \subseteq X \) contains its **minimal element** \( \min A \in A \), i.e. \( \min A \leq x \) for every \( x \in A \).

Further, define a **segment** of a well ordered set \( (X, \leq) \) to be a subset \( S \subseteq X \) such that for every \( a \in S \), we also have \( \{m, a\} := \{x \in X : x < a\} \subseteq S \). In addition, \( S \) is a **proper segment** of \( (X, \leq) \) if \( X \setminus S \) is non-empty.

**Theorem 1 (Zermelo’s Well Ordering Principle).** Every non-empty set \( X \) can be well ordered.

**Proof.** Let \( f \) be a function satisfying (1). Consider the family

\[
    \mathcal{F} := \{(W_\alpha, \leq_\alpha), \ \alpha \in I\} \tag{2}
\]

of all well ordered sets \( (W_\alpha, \leq_\alpha) \) satisfying \( m := f(X) = \min_\alpha W_\alpha \) – the minimal element of \( W_\alpha \) with respect to the ordering \( \leq_\alpha \), and

\[
    f(X \setminus S) = \min_\alpha (W_\alpha \setminus S) \in W_\alpha \setminus S \quad \text{for every proper segment} \quad S \quad \text{of} \quad (W_\alpha, \leq_\alpha). \tag{P}
\]

This means that \( f(X \setminus S) \) is the first subsequent element in \( (W_\alpha, \leq_\alpha) \) following \( S \). The equality in (P) can be rewritten in the form

\[
    S = [m, a)_\alpha := \{x \in W_\alpha : x < a\}, \quad \text{where} \quad a := f(X \setminus S) \in W_\alpha \setminus S. \tag{P1}
\]

Indeed, since \( S \) is a segment in \( (W_\alpha, \leq_\alpha) \), the inequality \( a <_\alpha x \) for \( x \in S \) brings to a contradiction: \( a \in [m, x)_\alpha \subseteq S \), whereas \( a \notin S \). Therefore, \( x <_\alpha a \) for all \( x \in S \), i.e. \( S \subseteq [m, a)_\alpha \).

On the other hand, from \( a = \min_\alpha (W_\alpha \setminus S) \) it follows \( [m, a)_\alpha \subseteq S \). Hence we have \( S = [m, a)_\alpha \).
Let well ordered sets \((W_1, \leq_1)\) and \((W_2, \leq_2)\) satisfy (P). Consider the common segments \(S\) of both these ordered sets, such that the ordering \(\leq_1\) agrees with \(\leq_2\) on \(S\). Then the union \(S_0\) of all common segments is the maximal (by inclusion) common segment, and \(S_0 \subseteq W_1 \cap W_2\). We claim that \(S_0\) coincides with at least one of the sets \(W_1\) or \(W_2\). Indeed, suppose otherwise. Then \(S_0\) is a proper segment in each of these two ordered sets, and from (P) it follows

\[
a_0 := f(X \setminus S_0) = \min_k(W_k \setminus S_0) \in W_k \setminus S_0 \quad \text{for} \quad k = 1, 2, \quad \text{hence} \quad a_0 \in (W_1 \cap W_2) \setminus S_0.
\]

By virtue of (P1), \(S_0 = [m, a_0]_1 = [m, a_0]_2\). Then \(S_0 \cup \{a_0\}\) is a common segment of \((W_1, \leq_1)\) and \((W_2, \leq_2)\), in contradiction with the maximality of \(S_0\).

The previous argument shows that for every two well ordered sets in (2), one is the extension of another, with the same ordering on the smaller set, which is a segment in the larger set. Note that if \(x \in W_{\alpha_1}\) and \(y \in W_{\alpha_2}\), then both \(x, y \in W_{\alpha}\) – the largest of \(W_{\alpha_1}\) and \(W_{\alpha_2}\). This allows us to define the chain \((X_0, \leq)\), where

\[
X_0 := \bigcup_{\alpha \in I} W_{\alpha}, \quad \text{and} \quad x \leq y \quad \text{if and only if} \quad x, y \in W_{\alpha} \quad \text{and} \quad x \leq_\alpha y \quad \text{for some} \quad \alpha. \quad (3)
\]

Further, we claim that \((X_0, \leq)\) is well ordered. Let \(A\) be a non-empty subset of \(X_0\). Then \(A \cap W_\beta\) is non-empty for some \(\beta\). Since \((W_\beta, \leq_\beta)\) is well ordered, there exists \(a := \min_\beta(A \cap W_\beta) \in A \cap W_\beta \subseteq A\). For the proof of our claim, it suffices to show that \(a = \min A\), i.e. \(a \leq x\) for every \(x \in A\), which in turn means that if \(a, x \in A \cap W_\alpha\), then \(a \leq_\alpha x\). If \(x \in A \cap W_\alpha \cap W_\beta\), this is true because the ordering \(\leq_\alpha\) agrees with \(\leq_\beta\), and \(a\) is the minimal element for a larger set \(A \cap W_\beta\). In the remaining case \(x \in A \cap (W_\alpha \setminus W_\beta)\), the set \(W_\beta\) is a proper segment of \((W_\alpha, \leq_\alpha)\). By virtue of (P1) and (P), we have

\[
a \in W_\beta = [m, b]_\alpha, \quad \text{where} \quad b := f(X \setminus W_\beta) = \min_\alpha(W_\alpha \setminus W_\beta),
\]

so that \(a \leq_\beta b \leq_\alpha x\). Thus we have \(a \leq x\) for every \(x \in A\), i.e. \(a = \min A\), and \((X_0, \leq)\) is well ordered.

If \(S\) is a proper segment of \((X_0, \leq)\), then the set \(W_\alpha \setminus S\) is non-empty for some \(\alpha\), and \(S\) is a proper segment of \((W_\alpha, \leq_\alpha)\) for some \(\alpha\). According to (P), we have

\[
a := f(X \setminus S) = \min_\alpha(W_\alpha \setminus S) = \min(X_0 \setminus S). \quad (4)
\]

Here the last equality follows from the facts that (i) the ordering \(\leq\) agrees with \(\leq_\alpha\) on \(W_\alpha \setminus S\), and (ii) if \(x \in X_0 \setminus W_\alpha\), then \(x \in W_\beta \setminus W_\alpha\), where \(W_\alpha\) is a proper segment of \(W_\beta\), so that \(a \leq x\) (which is the same as \(a \leq_\beta x\)) holds true automatically.

The property (4) allows us to include \((X_0, \leq)\) into the family \(F\) in (2). Finally, it remains to note that \(X_0 = X\), because otherwise one can compose a larger well ordered set \((X_0 \cup \{a_0\}, \leq)\) by taking \(a_0 := f(X \setminus X_0) \notin X_0\) as the subsequent element following \(X_0\). This extended set also belongs to the family \(F\), which contradicts to the choice of \(X_0\) in (3). Theorem is proved.

**Remark 2.** In the opposite direction, the Axiom of Choice follows from the Well Ordering Principle, simply by taking \(f(A) := m(A)\) in (2).
Theorem 3 (The Hausdorff Maximal Principle). Every chain \((A, \leq)\) in a partially ordered set \((X, \leq)\) is contained in a maximal chain \((L, \leq)\). In particular, maximal chains exist, because one can always start with a single point set \(A := \{a\}\), which satisfies \(a \leq a\).

Proof. If \(A = X\), then there is nothing to prove. In the contrary case, the set \(X_0 := X \setminus A\) is non-empty, and by Theorem 1, there is an ordering \(\leq_0\) (which has no relation to \(\leq\)) such that \((X_0, \leq_0)\) is a well ordered set.

Denote \(m_0 := \min_0 X_0 \in X_0\) – the minimal element in \(X_0\) with respect to the ordering \(\leq_0\). There are two possible cases: (i) \(m_0\) is comparable with every element \(x \in A\), i.e. we have either \(m_0 \leq x\) or \(x \leq m_0\); and the contrary case (ii) \(m_0\) is not comparable with some of \(x \in A\). In other words, we have either (i) \((A \cup \{m_0\}, \leq)\) is a chain, or (ii) \((A \cup \{m_0\}, \leq)\) is not a chain.

Following the well ordering \(\leq_0\), we can proceed by induction, deciding for every \(x \in X_0\) whether or not it should be included into a chain \((L_x, \leq)\), which appears as an extension of the original chain \((A, \leq)\). At the initial step, for \(x = m_0\), we have either (i) \(L_{x_0} = A \cup \{m_0\}\) or (ii) \(L_{x_0} = A\). We denote

\[ L'_x := \bigcup \{L_y : y \in X_0, y <_0 x\}, \]

assuming that all \(L_y\) in this expression are already defined.

Let \(B \subseteq X_0\) be the set of all elements \(b\) such that for all \(x \in X_0\) satisfying \(m_0 \leq x \leq_0 b\), the chains \((L_x, \leq)\) are uniquely defined and satisfy

\[
(i) \quad L_x := L'_x \cup \{x\} \quad \text{if} \quad \left( L'_x \cup \{x\}, \leq \right) \quad \text{is a chain}, \quad \quad (ii) \quad L_x := L'_x \quad \text{otherwise.} \quad (5)
\]

We claim that \(B = X_0\). Indeed, otherwise \(X_0 \setminus B\) is non-empty, and \(\exists a_0 := \min_0(X_0 \setminus B) \in X_0 \setminus B\). Then \(L'_{a_0}\) is the union of \(L_y\) over \(y \in B\), hence one can uniquely define \(L_{a_0}\) according to (5), and we must have \(a_0 \in B\). This contradiction proves the claim.

Finally, let \(L\) be the union of \(L_x\) over \(x \in X_0\). We need to show that the arbitrary \(x, y \in L\) are comparable, i.e. \(x \leq y\) or \(y \leq x\). If at least one of \(x\) or \(y\) belongs to \(A\), this follows from \(A \subseteq A_x\) for all \(x \in X_0\). In the remaining case \(x, y \in L \setminus A\), we can assume for certainty that \(y <_0 x\). Then by construction in (5),

\[ y \in L_y \subseteq L'_x \subseteq L_x, \quad \text{and} \quad x \in L_x. \]

Since both \(x, y \in L_x\), they are comparable. Thus \((L, \leq)\) is a chain.

This chain is maximal, because if \(x \notin L\), then \(x \in X_0\) and \(x \notin L_x\). Then by virtue of (4), \(x\) is not comparable with some elements of \(L'_x \subseteq L\). Theorem is proved.