Math 8601: REAL ANALYSIS. Fall 2015
Some problems for Midterm Exam #1 on Wednesday, October 14.

You will have 50 minutes (10:10 am–11:00 am) to work on 5 problems, 2 of which will be selected from the following list. It is recommended to prepare solutions of take-home problems on separate pages, then you can just enclose them together with in-class problems without rewriting.

No books and electronic devices. You can use class notes, including Appendices and Solutions to Homeworks.

#1. Let $f$ be a real function on $\mathbb{R}$. The image and the inverse image of a subset $A \subseteq \mathbb{R}$ under $f$ are correspondingly

$$f(A) = \{ y : y = f(x) \text{ for some } x \in A \}, \quad f^{-1}(A) = \{ x : f(x) \in A \}.$$ 

Show that $f(f^{-1}(A)) \subseteq A \subseteq f^{-1}(f(A))$ for arbitrary $A \subseteq \mathbb{R}$. Give an example when $f^{-1}(f(A)) \neq A$.

*Hint.* You can use without proof the properties of $f^{-1}$ on p.4, and Proposition 0.23 on p.14.

#2. For $A, B \subseteq \mathbb{R}$, define the algebraic sum

$$A + B = \{ x \in \mathbb{R} : x = a + b \text{ for some } a \in A, b \in B \}.$$ 

Either prove or give a counterexample for each of the following statements.

(a) If $A$ is open, then $A + B$ is open.

(b) If both $A$ and $B$ are bounded, then $A + B$ is bounded.

(c) If both $A$ and $B$ are closed, then $A + B$ is closed.

(d) If both $A$ and $B$ are compact, then $A + B$ is compact.

#3. Let $\{I_k\}$ be a sequence of disjoint intervals, each of which has one of the form $I_k = (a_k, b_k)$, $(a_k, b_k]$, $[a_k, b_k)$, or $[a_k, b_k]$, with $0 \leq a_k < b_k \leq 1$, and let $\{J_j\}$ be another sequence of intervals, such that

$$\bigcup_{k} I_k \subseteq \bigcup_{j} J_j.$$ 

Show that $\sum_{k} |I_k| \leq \sum_{j} |J_j|$, where $| \cdot |$ denotes the length of interval.

Two more problems are from the textbook (exercises 30,31 on p.40).

#4. Let $E$ be a Lebesgue measurable set in $\mathbb{R}$ with $m(E) > 0$, then for any $\alpha < 1$ there is an open interval $I$ such that $m(E \cap I) > \alpha m(I)$.

#5. By taking $\alpha > 3/4$ in the previous problem, show that

$$E - E := \{ x - y : x, y \in E \}$$

contains the interval $\left( - \frac{m(I)}{2}, \frac{m(I)}{2} \right)$. 
