Math 8601: REAL ANALYSIS. Fall 2015
Problems for Midterm Exam #2 on Wednesday, November 18.

This Midterm will be based on the material from the textbook up to (including) Section 2.3. You will have 50 minutes (10:10 am–11:00 am) to work on 5 problems, 3 of which will be selected from the following list. It is recommended to prepare solutions of take-home problems on separate pages, then you can just enclose them together with in-class problems without rewriting. The remaining 2 problems will be included into the 6th Homework assignment.

No books. You can use class notes. Calculators are permitted, however, for full credit, you need to show step-by-step calculations.

#1 (Borel-Cantelli Lemma). Let \((X, \mathcal{M}, \mu)\) be a measure space, and \(\{A_n\}\) be a sequence of sets in \(\mathcal{M}\). Show that

\[
\text{if } \sum_{n=1}^{\infty} \mu(A_n) < \infty, \text{ then } \mu \left( \limsup_{n \to \infty} A_n \right) = 0, \quad \text{where } \limsup_{n \to \infty} A_n := \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n.
\]

#2. Let \(f(x, y)\) be a function defined on the unit square \(\{0 \leq x \leq 1, \ 0 \leq y \leq 1\}\), which is continuous in each variable separately. Show that \(f\) is a Borel measurable function of \((x, y)\).

#3. Let \(f(x)\) be a right continuous function on \(\mathbb{R}^1\), i.e.

\[
\lim_{x \to x_0, x > x_0} f(x) = f(x_0) \quad \text{for all } x_0 \in \mathbb{R}^1.
\]

(a). Show that \(f\) is measurable.

(b). Show that \(f\) is continuous a.e. in \(\mathbb{R}^1\).

#4 (Problem 21 on p.59). Let \((X, \mathcal{M}, \mu)\) be a measure space, and let \(f, f_1, f_2, \ldots\) be non-negative functions functions in \(L^1(\mu)\), such that \(f_n \to f\) a.e. as \(n \to \infty\). Show that

\[
\text{if } \int f_n \, d\mu \to \int f \, d\mu, \text{ then } \int |f_n - f| \, d\mu \to 0.
\]

#5 (compare with Problem 31(d) on p.60). Show that for \(a > 0\),

\[
\int_0^\infty e^{-ax} x^{-1} \sin x \, dx = \arctan \left( a^{-1} \right).
\]

Hint. Apply Theorem 2.27(b).