1 (Exercise 30 on p.40). Let $E$ be a Lebesgue measurable set in $\mathbb{R}^1$ with Lebesgue measure $m(E) > 0$. Show that for any $\alpha < 1$, there is an open interval $I = (a, b)$ such that $m(E \cap I) > \alpha m(I)$.

2. (Exercise 31 on p.40). By taking $\alpha = 3/4$ in the previous problem, show that $E \subseteq E := \{x - y : x, y \in E\}$ contains the interval $\left( -\frac{m(I)}{2}, \frac{m(I)}{2} \right)$. 

Hint: First show that 

$$ z \in E - E \iff E \cap (z + E) \neq \emptyset. $$

3. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $A_j$ be a sequence in $\mathcal{M}$ satisfying 

$$ \mu(A_j \Delta A_k) \leq 2^{-j} \quad \text{for all} \quad 1 \leq j \leq k. $$

Show that $\exists A \in \mathcal{M}$ such that 

$$ \mu(A_j \Delta A) \to 0 \quad \text{as} \quad j \to \infty. $$

Hint: One can take 

$$ A := \liminf_{j \to \infty} A_j := \bigcup_{n=1}^{\infty} \bigcap_{j=n}^{\infty} A_j. $$

4. (Problem 3 on p.48). Let $(X, \mathcal{M})$ be a measurable space, and let $f_n : X \to \mathbb{R}^1$ be a sequence of $\mathcal{M}$-measurable functions on $X$. Show that the set 

$$ E := \{x \in X : \exists \lim_{n \to \infty} f_n(x)\} $$

is $\mathcal{M}$-measurable.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Show that 

$$ d(f + g) \leq d(f) + d(g) \quad \text{for all measurable functions} \quad f, g \quad \text{on} \quad X, $$

where 

$$ d(f) := \inf_{\varepsilon > 0} \left[ \varepsilon + \mu\{x \in X : |f(x)| \geq \varepsilon\} \right]. $$