#1 (Problem 16 on p.52). Let \((X, \mathcal{M}, \mu)\) be a measure space, and let \(f \in L^+\) - the space of all measurable functions from \(X\) to \([0, \infty]\), with \(\int_X f \, d\mu < \infty\). Show that for every \(\varepsilon > 0\), there exists a set \(E \in \mathcal{M}\) such that
\[
\mu(E) < \infty \quad \text{and} \quad \int_E f \, d\mu > \int_X f \, d\mu - \varepsilon.
\]

#2. Let \(f \in L^1(\mathbb{R}^1)\). Show that
\[
\omega(h) := \sup_{|t| \leq h} \int \left| f(x + t) - f(x) \right| \, dx \to 0 \quad \text{as} \quad h \to 0^+.
\]

*Hint.* Use Theorem 2.26.

#3. Under assumptions of the previous problem, show that
\[
\int_{\mathbb{R}^1} |S_n(x) - S(x)| \, dx \leq \omega(1/n),
\]
where
\[
S_n(x) := \frac{1}{n} \sum_{j=0}^{n-1} f \left( x + \frac{j}{n} \right), \quad S(x) = \int_{x}^{x+1} f(t) \, dt.
\]

#4. Let \(f, f_1, f_2, \ldots, f_n, \ldots\) be measurable functions on a measure space \((X, \mathcal{M}, \mu)\) with \(\mu(X) = 1\), such that \(f_n(x) \to f(x)\) as \(n \to \infty\) for all \(x \in X\). Suppose that
\[
\int |f_n|^{1+\alpha} \, d\mu \leq C \quad \text{for all} \quad n, \quad \text{with some constants} \quad \alpha > 0 \quad \text{and} \quad C > 0.
\]
Show that
\[
\int |f_n - f| \, d\mu \to 0 \quad \text{as} \quad n \to \infty.
\]

#5. Show that any non-negative measurable function \(f(x)\) on a measurable space \((X, \mathcal{M})\) can be represented in the form
\[
f = \sum_{n=1}^{\infty} \frac{1}{n} \cdot I_{E_n}, \quad \text{where} \quad I_{E_n}(x) := \begin{cases} 1 & \text{if} \quad x \in E_n, \\ 0 & \text{if} \quad x \notin E_n. \end{cases}
\]

*Hint.* This representation is not unique. One can try to find the “maximal possible” \(E_1\), then \(E_2\), etc.