Appendix 1: Fourier Transforms

Definition 1. The Fourier transform of a function $f(x) \in L^1(\mathbb{R}^n)$ is

$$g(\omega) = F[f](\omega) := \int_{\mathbb{R}^n} e^{-i\omega x} f(x) \, dx.$$  \hspace{1cm} (1)

Here $x = (x_1, \ldots, x_n)$, $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n$,

$$e^{-i\omega x} := \cos \omega x - i \sin \omega x, \hspace{0.5cm} \omega x := \omega_1 x_1 + \cdots + \omega_n x_n.$$  

Since $|e^{-i\omega x}| = |f| \in L^1$, by Lebesgue’s Dominated Convergence Theorem we have $\lim_{\omega \to \omega_0} g(\omega) = g(\omega_0)$, i.e. $g = F[f]$ is continuous for every $f \in L^1$. Obviously, $F$ is also bounded as an operator from $L^1$ to $L^\infty$ with $\|F[f]\|_{\infty} \leq |f|_1$.

First we restrict $F$ to the Schwartz space $S \subset C^\infty(\mathbb{R}^n)$ of functions $f$ satisfying

$$\sup_{\mathbb{R}^n} |x^\alpha D^\beta f(x)| = \sup_{\mathbb{R}^n} |x^{\alpha_1} \cdots x^{\alpha_n} \frac{\partial^{\beta_1 + \cdots + \beta_n} f}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}| < \infty$$

for all multi-indices $\alpha, \beta \geq 0$. Since $(1 + |x|^2)^n$ is a polynomial, and $(1 + |x|^2)^{-n} \in L^1$, we also have

$$|(1 + |x|^2)^n x^\alpha D^\beta f| \leq C(\alpha, \beta) = \text{const} < \infty, \quad x^\alpha D^\beta f \in L^1, \quad \text{and} \quad F[x^\alpha D^\beta f] \in L^\infty.$$

Lemma 1. For $f \in S$, $g(\omega) = F[f](\omega)$, and all multi-indices $\alpha, \beta \geq 0$, we have:

(a) $\omega^\alpha g(\omega) = F[(-D)^\alpha f](\omega)$; \hspace{0.5cm} (b) $D^\beta g(\omega) = F[(-i x)^\beta f](\omega)$.

Proof. The property (a) follows by integration by parts, (b) – by differentiation of the equality (1). \hfill \Box

Corollary 1. If $f \in S$, then $g := F[f] \in S$.

Proof. By the previous lemma, $\omega^\alpha D^\beta g$ is a finite linear combination of $F[x^\mu D^\nu f]$ with multi-indices $\mu, \nu \geq 0$. Since all $x^\mu D^\nu f$ belong to $L^1$, we have $|\omega^\alpha D^\beta g| \leq C = \text{const} < \infty$. \hfill \Box

We also define the inverse Fourier transform of any function $g(\omega) \in S$ by the formula

$$f(x) = F^{-1}[g](x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\omega x} g(\omega) \, d\omega = (2\pi)^{-n} F[\overline{g}].$$  \hspace{1cm} (2)

From Corollary 1 and the last equality in (2) it follows that if $g \in S$, then $F^{-1}[g] \in S$. We will show that indeed, $F^{-1}$ is the inverse operator of $F$ on $S$ (equalities (10) in Theorem 2 below). In the following example, we check these equalities for $f = \varphi := e^{-\frac{x^2}{2}}$.

Example 1. We will find the Fourier transform of the function $\varphi(x) = e^{-\frac{x^2}{2}}$ on $\mathbb{R}^1$. Since $\varphi$ is an even function, we have

$$g(\omega) = F[\varphi](\omega) = \int_{\mathbb{R}^1} \cos \omega x \cdot e^{-\frac{x^2}{2}} \, dx.$$
Using polar coordinates, we get

\[ g^2(0) = \int_{\mathbb{R}^1} e^{-\frac{x^2}{2}} \, dx \cdot \int_{\mathbb{R}^1} e^{-\frac{y^2}{2}} \, dy = \iint_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}} \, dx \, dy = \int_0^{2\pi} \int_0^\infty e^{-\frac{r^2}{2}} \, r \, dr \, d\theta = 2\pi, \]

hence \( g(0) = \sqrt{2\pi} \). Further,

\[ g'(\omega) = -\int_{\mathbb{R}^1} \sin \omega x \cdot xe^{-\frac{x^2}{2}} \, dx = \int_{\mathbb{R}^1} \sin \omega x \cdot d\varphi(x) = -\int \omega \cos \omega x \cdot \varphi(x) \, dx = -\omega g(\omega), \]

\[ (\ln g)' = -\omega, \quad \ln g = \text{const} - \frac{\omega^2}{2}, \]

and since \( g(0) = \sqrt{2\pi} \),

\[ g(\omega) := F[\varphi](\omega) = \text{const} \cdot e^{-\frac{\omega^2}{2}} = \sqrt{2\pi} \cdot e^{-\frac{\omega^2}{2}}. \]

We will use the same notation \( \varphi \) for the function

\[ \varphi(x) = e^{-\frac{x^2}{2}} = e^{-\frac{1}{2} \sum x_k^2} \quad \text{on} \quad \mathbb{R}^n. \]

Its Fourier transform is represented as the product:

\[ g(\omega) = \int_{\mathbb{R}^n} e^{-i\omega x} e^{-\frac{x^2}{2}} \, dx = \prod_{k=1}^n \int_{\mathbb{R}^1} e^{-i\omega_k x_k} e^{-\frac{x_k^2}{2}} \, dx_k = \prod_{k=1}^n F[\varphi](\omega_k), \]

and by the above formula,

\[ F[\varphi](\omega) = F\left[ e^{-\frac{x^2}{2}} \right](\omega) = \prod_{k=1}^n \left( \sqrt{2\pi} \cdot e^{-\frac{\omega^2}{4}} \right) = (2\pi)^\frac{n}{2} e^{-\omega^2} = (2\pi)^\frac{n}{2} \varphi(\omega). \quad (3) \]

The previous calculations remain the same if we replace \( i \) by \( -i \). Therefore,

\[ F^{-1}[\varphi](x) = (2\pi)^{-n} F[\varphi](x) = (2\pi)^{-\frac{n}{2}} \varphi(x), \]

and

\[ F[F^{-1}[\varphi]](x) = (2\pi)^{-\frac{n}{2}} F[\varphi](x) = \varphi(x), \quad F^{-1}[F[\varphi]](x) = (2\pi)^{\frac{n}{2}} F^{-1}[\varphi](x) = \varphi(x). \quad (4) \]

**Theorem 1.** For any constants \( k > 0 \) and \( h \in \mathbb{R}^n \), operators \( F \) and \( F^{-1} \) defined by formulas (1) and (2) on \( S \), satisfy the equalities

\[ F[f(kx)](\omega) = k^{-n} F[f(x)](k^{-1} \omega), \quad F^{-1}[g(k\omega)](x) = k^{-n} F^{-1}[g(\omega)](k^{-1} x), \quad (5) \]

\[ F[f(x+h)](\omega) = e^{i\omega h} F[f(x)](\omega), \quad F^{-1}[g(\omega + h)](x) = e^{-i\omega x} F^{-1}[g(\omega)](x), \quad (6) \]

\[ F[e^{i\omega x} f(x)](\omega) = F[f(x)](\omega - h), \quad F^{-1}[e^{ih\omega} g(\omega)](x) = F^{-1}[g(\omega)](x + h), \quad (7) \]

\[ F[f*g] = F[f] \cdot F[g]. \quad (8) \]
Proof of (5)–(7) is easy to obtain by changing the variables. The equality (8) follows from Fubini’s theorem:

\[ F[f \ast g](\omega) = \int e^{-i\omega x} \left[ \int f(x-t)g(t) \, dt \right] \, dx \]

\[ = \int e^{-i\omega t}g(t) \left[ \int e^{-i\omega(x-t)}f(x-t) \, dx \right] \, dt = F[f](\omega) \cdot \int e^{-i\omega t}g(t) \, dt = F[f](\omega) \cdot F[g](\omega). \]

Note that the complex-valued functions \( f : \mathbb{R}^n \to C \) in \( L^2(\mathbb{R}^n) \) compose a Hilbert space with the inner (or scalar) product

\[ \langle f, g \rangle := \int f \overline{g} \, dx. \]

**Theorem 2.** For all \( f, g \in S \), we have

\[ \langle F[f], g \rangle = \int_{\mathbb{R}^n} F[f](\omega) \cdot \overline{g(\omega)} \, d\omega = (2\pi)^n \langle f, F^{-1}[g] \rangle. \]

Moreover,

\[ F^{-1}[F[f]] = f, \quad F[F^{-1}[f]] = f, \]

and the following Plancherel equalities hold true:

\[ \| F[f] \|_2^2 = (2\pi)^n \| f \|_2^2, \quad \| f \|_2^2 = (2\pi)^n \| F^{-1}[f] \|_2^2. \]

**Proof.** The equality (9) follows from Fubini’s theorem:

\[ \langle F[f], g \rangle = \int \left[ \int e^{-i\omega x} f(x) \, dx \right] \overline{g(\omega)} \, d\omega = \int f(x) \left[ \int e^{i\omega x} g(\omega) \, d\omega \right] \, dx = (2\pi)^n \langle f, F^{-1}[g] \rangle. \]

Further, denote by \( S_0 \) the set of all functions \( f \in S \) satisfying (10). By Theorem 1 with \( g := F[f] \), we have

\[ F[f(kx)](\omega) = k^{-n}g(k^{-1}\omega), \quad F^{-1}[k^{-n}g(k^{-1}\omega)](x) = f(kx); \]

\[ F[f(x+h)](\omega) = e^{i\omega h}g(\omega), \quad F^{-1}[e^{i\omega h}g(\omega)](x) = f(x+h). \]

In other words, if \( f \in S_0 \), then \( f(kx) \) and \( f(x+h) \) satisfy the first equality in (10). The second equality follows from the relation \( F^{-1}[f] = (2\pi)^{-n}F[F] \). Therefore, from \( f \in S_0 \) it follows that \( f(kx), f(x+h) \in S_0 \).

By virtue of (4), we know that \( \varphi(x) = e^{-\frac{x^2}{2}} \in S_0 \). Then

\[ K(x) := (2\pi)^{-\frac{n}{2}} \varphi(x), \quad K_\varepsilon(x) := e^{-\varepsilon x}K(\varepsilon^{-1}x), \quad \text{and} \quad K_\varepsilon(x-t) \]

belong to \( S_0 \) for all \( \varepsilon > 0 \) and \( t \in \mathbb{R}^n \). It is easy to verify that

\[ f_\varepsilon(x) := (f \ast K_\varepsilon)(x) = \int_{\mathbb{R}^n} f(t) K_\varepsilon(x-t) \, dt = \int_{\mathbb{R}^n} f(x-\varepsilon y) K(y) \, dy \]
belongs to \( S \) for all \( f \in S \) and \( \varepsilon > 0 \). In addition,
\[
F^{-1}[F[f_{\varepsilon}]] = \int_{\mathbb{R}^n} f(t) F^{-1}[F[K^\varepsilon(x-t)]] \, dt = \int_{\mathbb{R}^n} f(t) K^\varepsilon(x-t) \, dt = f_{\varepsilon},
\]
and similarly, \( F[F^{-1}[f_{\varepsilon}]] = f_{\varepsilon} \). This means that in fact we have \( f_{\varepsilon} \in S_0 \).

By our choice of constants, we have \( \int K(y) \, dy = 1 \). Hence
\[
(f_{\varepsilon} - f)(x) = \int_{\mathbb{R}^n} \left[ f(x - \varepsilon y) - f(x) \right] \cdot K(y) \, dy.
\]

Using Minkowski’s integral inequality, we estimate the \( L^2 \)-norm as follows:
\[
||f_{\varepsilon} - f||_2 \leq \int_{\mathbb{R}^n} ||f(x - \varepsilon y) - f(x)||_2 \cdot K(y) \, dy.
\]

We have \( ||f(x - h) - f(x)||_2 \to 0 \) as \( h \to 0 \), even for \( f \in L^2 \). Then by the Dominated Convergence Theorem, \( ||f_{\varepsilon} - f||_2 \to 0 \) as \( \varepsilon \to 0 \).

Finally, if \( f \in S \) satisfies (10), then (11) follows from (9) with \( g := F[f] \). The previous argument shows that these equalities hold true on a family \( S_0 \subseteq S \), which is dense in \( S_0 \) with respect to the \( L^2 \)-norm. By standard approximation, these properties are extended to the whole class \( S \), i.e. \( S_0 = S \).

The Plancherel equalities allow to define Fourier transforms \( F \) and \( F^{-1} \) for functions \( f \in L^2 \) as limits in \( L^2 \):
\[
F[f] := \lim_{n \to \infty} F[f_n], \quad F^{-1}[f] := \lim_{n \to \infty} F^{-1}[f_n], \quad \text{where} \quad f = \lim_{n \to \infty} f_n, \quad f_n \in S.
\]

Then “by continuity”, all the equalities (5)–(7) and (9)–(11) also hold true for functions in \( L^2 \).

**Example 2.** The Fourier transform of the function \( f(x) := e^{-k|x|} \) on \( \mathbb{R}^1 \), where \( k = \text{const} > 0 \), is
\[
g(\omega) := F[f](\omega) = \int_{\mathbb{R}^1} e^{-i\omega x - k|x|} \, dx = 2 \cdot \text{Re} \int_{0}^{\infty} e^{-(k+i\omega)x} \, dx = 2 \cdot \text{Re} \frac{1}{k + i\omega} = \frac{2k}{k^2 + \omega^2}.
\]

Since \( g \) is an even function, we also have
\[
F[g](\omega) := \int_{\mathbb{R}^1} e^{-i\omega x} g(x) \, dx = \int_{\mathbb{R}^1} e^{i\omega x} g(x) \, dx = 2\pi \cdot F^{-1}[g](\omega) = 2\pi \cdot f(\omega),
\]
and
\[
F^{-1}\left[\frac{k}{k^2 + x^2}\right](\omega) = \frac{1}{2} \cdot F[g](\omega) = \pi \cdot e^{-k|\omega|}.
\]

**Definition 2.** For \( n = 0, 1, 2, \ldots \), the **Hermite polynomials** are defined as
\[
H_n(x) := (-1)^n e^{x^2} \left(e^{-x^2}\right)^{(n)},
\]
so that \( H_0 = 1, H_1 = 2x \), etc. The corresponding **Hermite functions** \( \varphi_n(x) := H_n(x) e^{-x^2/2} \).
In particular, \( \varphi_0(x) = e^{-\frac{x^2}{2}} = \varphi(x) \) in Example 1. All functions \( \varphi_n \) belong to the Schwartz space \( S \).

Using integration by parts, it is easy to check that the system \( \{\varphi_n\} \) is orthogonal in \( L^2 := L^2(\mathbb{R}^1) \):

\[
\langle \varphi_m, \varphi_n \rangle := \int_{\mathbb{R}^1} \varphi_m(x)\varphi_n(x)dx = \int_{\mathbb{R}^1} H_m H_n e^{-x^2}dx = 0 \quad \text{for} \ m \neq n.
\]

**Theorem 3.** The system of Hermite functions \( \{\varphi_n\} \) is complete in \( L^2 \), i.e. from \( f \in L^2 \) and \( \langle \varphi_n, f \rangle = 0 \) for all \( n \) it follows \( f = 0 \) a.e.

**Proof.** The assumption \( \langle \varphi_n, f \rangle = 0 \) for all \( n \) is equivalent to \( \langle x^n \varphi, f \rangle = 0 \) for all \( n \), because every \( x^n \) is a linear combination of \( H_k \), \( k \leq n \), and correspondingly, \( x^n \varphi \) is a linear combination of \( H_k \varphi = \varphi_k \), \( k \leq n \). Consider the Fourier transform

\[
g(\omega) := F[\varphi f](\omega) = \int_{\mathbb{R}^1} e^{-i\omega x - \frac{x^2}{2}} f(x) dx.
\]

This integral is well defined for complex \( \omega = \omega_1 + i\omega_2 \), and \( g(\omega) \) is analytic in the whole complex plane \( C \).

By our assumptions, all the derivatives

\[
g^{(n)}(0) = \int_{\mathbb{R}^1} (-ix)^n e^{-\frac{x^2}{2}} f(x) dx = (-i)^n \langle x^n \varphi, f \rangle = 0.
\]

By uniqueness for analytic functions, we must have \( g \equiv 0 \). Finally, from the Plancherel equality it follows

\[
2\pi \cdot ||\varphi f||_2^2 = ||g||_2^2 = 0,
\]

so that \( f = 0 \) a.e. \( \square \)

**Theorem 4.** The Hermite functions \( \varphi_n \) are eigenfunctions of the Fourier transform:

\[
F[\varphi_n] = c_n \varphi_n, \quad \text{where} \quad c_n := (-i)^n \sqrt{2\pi} \quad \text{for} \ n = 0, 1, 2, \ldots.
\]

**Proof.** We know that this property holds true for \( n = 0 \) with \( c_0 := \sqrt{2\pi} \). Moreover,

\[
(x - D)\varphi_k = (-1)^k(x - D) \left[ e^{\frac{x^2}{2}} (e^{-x^2})^{(k)} \right] = (-1)^{k+1} e^{\frac{x^2}{2}} (e^{-x^2})^{(k+1)} = \varphi_{k+1},
\]

so that by induction, \( \varphi_n = (x - D)^n \varphi \) for all \( n = 0, 1, 2, \ldots \). Note that by Lemma 1,

\[
F[(x - D)f] = -i F[(-i)(D - x)f] = -i (\omega - D) F[f] \quad \text{for} \ f \in S.
\]

Therefore,

\[
F[\varphi_n] = F[(x - D)^n \varphi] = (-i)^n (\omega - D)^n F[\varphi] = (-i)^n \sqrt{2\pi} \cdot (\omega - D)^n \varphi = (-i)^n \sqrt{2\pi} \cdot \varphi_n.
\]

Theorem is proved. \( \square \)

At the conclusion, we prove a few relations between the Fourier operator \( F \), the differential operator \( L := D^2 - x^2 \), and the Hermite functions \( \varphi_n := H_n \varphi \).
Theorem 5. (a). The Fourier operator \( F \) is commutative with \( L := D^2 - x^2 \) on \( S \):
\[
F[Lf] = LF[f] \quad \text{for} \quad f \in S.
\] (13)

In particular, if \( Lf := f'' - x^2f = 0 \), then \( g(\omega) := F[f](\omega) \) also satisfies \( Lg := g'' - \omega^2g = 0 \).

(b). The Hermite functions \( \varphi_n := H_n\varphi \) are eigenfunctions of \( L \):
\[
L \varphi_n := \varphi_n'' - x^2 \varphi_n = \lambda_n \varphi_n \quad \text{with} \quad \lambda_n := -(2n + 1) \quad \text{for} \quad n = 0, 1, 2, \ldots.
\] (14)

(c). The Hermite polynomials \( H_n \) satisfy the \textbf{Hermite equation}
\[
y'' - 2xy' = \mu y \quad \text{with} \quad \mu = 2n \quad \text{for} \quad n = 0, 1, 2, \ldots.
\] (15)

\textbf{Proof. (a).} By Lemma 1, the functions \( f \) and \( g := F[f] \) in \( S \) satisfy
\[
F[D^2f] = -F[(iD)^2f] = -\omega^2g, \quad F[-x^2f] = F[(ix)^2f] = D^2g.
\]
From these relations, the equality (13) follows:
\[
F[Lf] = F[D^2f - x^2f] = -\omega^2g + D^2g = Lg = LF[f].
\]

(b) and (c). We will try to find polynomials \( P_n \) of degree \( n \) (eventually \( P_n = \text{const} \cdot H_n \)) such that \( \psi_n := P_n\varphi \) satisfy \( L\psi_n = \psi_n'' - x^2\psi_n = \lambda \psi_n \) with a constant \( \lambda \) (depending on \( n \)). Since \( \varphi(x) := e^{-x^2/2} \) satisfies \( \varphi' = -x\varphi, \varphi'' = (x^2 - 1)\varphi \), we get
\[
L\psi_n = P_n''\varphi + 2P_n'\varphi' + P_n\varphi'' - x^2P_n\varphi = (P_n'' - 2xP_n' - P_n)\varphi = \lambda P_n\varphi.
\]
Here \( P_n = \sum_{k=0}^{n} a_k x^k, \ a_n \neq 0 \). Comparing the coefficients of \( x^n \) in both sides, we see that the equality is only possible if \( \lambda = \lambda_n := -(2n + 1) \). One can select \( a_n \neq 0 \) in an arbitrary way, and then the remaining coefficients \( a_k \) are uniquely defined by a standard recurrent procedure.

From the equalities \( L\psi_k = \lambda_k\psi_k \) it follows
\[
(\psi_m\psi_n - \psi_n\psi_m)' = \psi_m''\psi_n - \psi_n''\psi_m = (\lambda_m - \lambda_n)\psi_m\psi_n.
\]
Integrating over \( \mathbb{R}^1 \) yields
\[
0 = (\lambda_m - \lambda_n) \cdot \langle \psi_m, \psi_n \rangle, \quad \text{so that} \quad \{\psi_n\} \text{ is an orthogonal system in } L^2.
\]
Note that both \( \{\varphi_n := H_n\varphi\} \) and \( \{\psi_n := P_n\varphi\} \) can be obtained by orthogonalization of \( \{x^n\varphi\} \), i.e. \( \langle \varphi_n, x^k\varphi \rangle = \langle \psi_n, x^k\varphi \rangle = 0 \) for all \( k \leq n - 1 \). From this observation it follows that \( \varphi_n = \text{const} \cdot \psi_n \) and \( H_n = \text{const} \cdot P_n \). Finally, since \( L\psi_n = \lambda_n\psi_n \) and \( P_n'' - 2xP_n' + 2nP_n = 0 \), the functions \( \varphi_n := H_n\varphi \) satisfy (14), and \( y = H_n \) satisfy (15). Theorem is proved. 

\( \square \)