
**Problem 1.** Show that for any two Borel measurable sets \( E_1, E_2 \subseteq \mathbb{R}^1 \) with finite Borel measure, the convolution

\[
f(x) = (I_{E_1} * I_{E_2})(x) := \int_{\mathbb{R}^1} I_{E_1}(x-y) I_{E_2}(y) \, dy,
\]

where \( I_E(y) = 1 \) if \( y \in E \), and \( I_E(y) = 0 \) if \( y \notin E \), is continuous on \( \mathbb{R}^1 \).

**Proof.** Both functions \( I_{E_1} \) and \( I_{E_2} \) belong to \( L^1 := L^1(\mathbb{R}^1) \), so that they can be approximated in \( L^1 \) by continuous functions with compact support (Theorem 2.41 on p.70). For a fixed \( \varepsilon > 0 \), choose such a function \( g \) for which the norm in \( L^1 \), \( \| I_{E_1} - g \|_1 \leq \varepsilon/2 \). Then the function \( f_1 := g * I_{E_2} \) satisfies

\[
| (f - f_1)(x) | = | (I_{E_1} - g) * I_{E_2}(x) | \leq \| I_{E_1} - g \|_1 \leq \varepsilon/2, \quad \forall x \in \mathbb{R}^1.
\]

Hence

\[
|f(x + h) - f(x)| \leq |f_1(x + h) - f_1(x)| + 2 \cdot \sup |f - f_1| \leq |f_1(x + h) - f_1(x)| + \varepsilon, \quad \forall x, h \in \mathbb{R}^1.
\]

By Theorem 2.27 (a) on p.56, the function \( f_1 \) is continuous on \( \mathbb{R}^1 \). Therefore,

\[
\lim_{h \to 0} \sup |f(x + h) - f(x)| \leq \varepsilon, \quad \forall x \in \mathbb{R}^1.
\]

Since \( \varepsilon > 0 \) is arbitrary, \( f \) is continuous.

**Problem 2. (a).** Let \( K \) be a nonempty closed set in \( \mathbb{R}^n \). Show that for every \( \varepsilon > 0 \), the sets

\[
K^\varepsilon := \{ x \in \mathbb{R}^n : \text{dist}(x, K) \leq \varepsilon \}
\]

are closed in \( \mathbb{R}^n \).

(b). Let \( K_j \) be a sequence of nonempty compact sets in \( \mathbb{R}^n \), such that \( K_1 \supseteq K_2 \supseteq \cdots \). Set \( K := \bigcap_j K_j \). Show that for every \( \varepsilon > 0 \), there exists a constant \( N = N(\varepsilon) \) such that \( K_j \subseteq K^\varepsilon \), \( \forall j \geq N \).

**Proof. (a).** Taking the infimum over \( y \in K \) in the inequality \( |x_1 - y| \leq |x_2 - y| + |x_1 - x_2| \), we see that the distance function \( d(x) := \text{dist}(x, K) := \inf \{|x - y| : y \in K\} \) satisfies

\[
d(x_1) \leq d(x_2) + |x_1 - x_2|.
\]

By symmetry, we always have

\[
|d(x_1) - d(x_2)| \leq |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n.
\]

In particular, \( d(x) \) is continuous, and \( K^\varepsilon = d^{-1}(0, \varepsilon) \) is closed.
(b). The compact set

\[ F := K_1 \cap \{ d(x) \geq \varepsilon \} \subseteq K^c = \bigcup_{j=1}^{\infty} K_j^c, \]

where \( K_j^c \) are open. Therefore, \( F \subseteq K_N^c \) for some \( N \), and

\[ K_j \subseteq K_N \subseteq (F^c) \cap K_1 \subseteq \{ d(x) < \varepsilon \} \subseteq K^c, \quad \forall j \geq N. \]

**Problem 3.** Let \( f, f_1, f_2, \ldots \) be real measurable functions on \( \mathbb{R} \), such that \( f_n \to f \) almost everywhere (a.e.) as \( n \to \infty \), and

\[ \int_{\mathbb{R}} f(x) \, dx = 1, \quad \int_{\mathbb{R}} f_n(x) \, dx = 1, \quad \text{and} \quad f_n \geq 0 \quad \text{for all} \quad n. \]

(a). Show that \( f_n \to f \) in \( L^1(\mathbb{R}) \) as \( n \to \infty \).

(b). Show that (a) may fail if the assumption \( \int f(x) \, dx = 1 \) is dropped.

(c). Show that (a) may fail if the assumption \( f_n \geq 0 \) is dropped.

**Proof.** (a). We have \( 0 \leq g_n := \min\{f, f_n\} \leq f \in L^1 \), and \( g_n \to f \) a.e. By the Dominated Convergence Theorem,

\[ \int_{\mathbb{R}} g_n \, dx \to \int_{\mathbb{R}} f \, dx = 1 \quad \text{as} \quad n \to \infty. \]

Moreover, it is easy to see that \( |f_n - f| = f_n + f - 2g_n \). Therefore,

\[ \int_{\mathbb{R}} |f_n - f| \, dx = \int_{\mathbb{R}} f_n \, dx + \int_{\mathbb{R}} f \, dx - 2 \int_{\mathbb{R}} g_n \, dx = 2 - 2 \int_{\mathbb{R}} g_n \, dx \to 0 \quad \text{as} \quad n \to \infty. \]

(b). \( f_n := I_{(n,n+1)} \to f \equiv 0 \) a.e., but not in \( L^1 \).

(c). \( f_n := I_{(0,1)} - I_{(n,n+1)} + I_{(n+1,n+2)} \to f := I_{(0,1)} \) a.e., but not in \( L^1 \).

**Problem 4.** Let \( f(x) \) be a continuous function on \([-1,1]\), such that

\[ \int_{-1}^{1} x^k f(x) \, dx = 0 \quad \text{for all} \quad k = 0, 1, 2, \ldots. \]

Show that \( f \equiv 0 \) on \([-1,1]\).

**Proof.** From linearity it follows that

\[ \int_{-1}^{1} pf \, dx = 0 \quad \text{for every polynomial} \quad p. \]

By the Weierstrass theorem, \( f(x) \) can be approximated by a sequence of polynomials \( p_n \) uniformly on \([-1,1]\). Then we must have

\[ \int_{-1}^{1} f^2 \, dx = \lim_{n \to \infty} \int_{-1}^{1} p_n f \, dx = 0, \quad \text{and} \quad f \equiv 0 \quad \text{on} \quad [-1,1]. \]