Math 8602: REAL ANALYSIS. Spring 2016

Homework #5. Problems and Solutions.

#1. Let $f$ be a function in $L^1(\mathbb{R}^1)$. Show that

$$\int_{\mathbb{R}^1} f(x) \sin(\omega x) \, dx \to 0 \text{ as } \omega \to \infty.$$

**Proof.** This problem is very similar to Problem 3 on Final Exam in the previous semester. By Theorem 2.26, every function $f \in L^1(\mathbb{R}^1)$ can be approximated in $L^1(\mathbb{R}^1)$ by functions $g \in C_0(\mathbb{R}^1)$ — continuous functions with compact support. In turn, by the Dominated Convergence Theorem, every function $g \in C_0(\mathbb{R}^1)$ can be approximated in $L^1(\mathbb{R}^1)$ by functions

$$g_h(x) := \frac{1}{h} \int_x^{x+h} g(y) \, dy \in (C^1 \cap C_0)(\mathbb{R}^1),$$

i.e. the $L^1$-norms $\|g_h - g\|_1 \to 0$ as $h \searrow 0$. Therefore, $\forall f \in L^1(\mathbb{R}^1)$ and $\forall \varepsilon > 0$, $\exists g_h \in (C^1 \cap C_0)(\mathbb{R}^1)$ with $\|g_h - f\|_1 \leq \varepsilon$. We can write

$$I(\omega) := \int_{\mathbb{R}^1} f(x) \sin(\omega x) \, dx = I_1(\omega) + I_2(\omega),$$

where

$$I_1(\omega) := \int_{\mathbb{R}^1} [f(x) - g_h(x)] \sin(\omega x) \, dx, \quad |I_1(\omega)| \leq \|g_h - f\|_1 \leq \varepsilon,$$

$$I_2(\omega) := \int_{\mathbb{R}^1} g_h(x) \sin(\omega x) \, dx, \quad |I_2(\omega)| \leq \frac{1}{\omega} \left( \int_{\mathbb{R}^1} \left| g_h'(x) \cos(\omega x) \right| \, dx \right) \leq \frac{1}{\omega} \|g_h\|_1 \to 0 \text{ as } \omega \to \infty.$$ 

Then $\limsup_{\omega \to \infty} |I(\omega)| \leq \varepsilon$, and since $\varepsilon > 0$ can be taken arbitrarily small, we get $I(\omega) \to 0$ as $\omega \to \infty$.

#2. Let $f(x) \in L^1_{loc}(\mathbb{R})$ and

$$f\left(\frac{x + y}{2}\right) \leq \frac{f(x) + f(y)}{2} \quad \text{for all } x, y \in \mathbb{R}.$$ 

Show that $f$ is convex on $\mathbb{R}$.

**Proof.** This statement is true under a more general assumption that $|f| < \infty$ a.e. The convexity of $f$ means that a portion of the graph of $y = f(x)$ between two arbitrary point $x_1$ and $x_2$ in $\mathbb{R}$ lies below the segment connecting the point $(x_1, f(x_1))$ and $(x_2, f(x_2))$ in $\mathbb{R}^2$. By a linear transform, the proof of this fact is reduced to the case $x_1 = -1$, $x_2 = 1$, and $f(-1) = f(1) = 0$; in this case we must have $f(x) \leq 0$ on $[-1, 1]$.

Suppose otherwise, i.e. $f(x_0) \geq a = \text{const} > 0$ for some $x_0 \in (-1, 1)$. Take a small $h_0 > 0$, such that $[x_0 - h_0, x_0 + h_0] \subseteq [-1, 1]$. By our assumptions,

$$0 < a \leq f(x_0) \leq \frac{f(x_0 + h) + f(x_0 - h)}{2}, \quad \forall h \in [-h_0, h_0].$$

For such $h$, either $f(x_0 + h) \geq a$ or $f(x_0 - h) \geq a$. In other words,

$$[-h_0, h_0] = A \cup (-A), \quad \text{where } A := \{h \in [-h_0, h_0] : f(x_0 + h) \geq a\}.$$ 

Then the set $E(a) := [-1, 1] \cap \{f \geq a > 0\}$ contains $x_0 + A$, and its Lebesgue measure

$$m(E(a)) \geq m(A) = \frac{1}{2} \cdot (m(A) + m(-A)) \geq \frac{1}{2} \cdot m(A) = \frac{1}{2} \cdot m([-h_0, h_0]) = h_0 > 0.$$
On the other hand,

\[ E(a) = E^-(a) \cup E^+(a), \quad \text{where} \quad E^-(a) := E(a) \cap [-1,0], \quad E^+(a) := E(a) \cap [0,1], \]

so that \( m(E^-(a)) + m(E^+(a)) = m(E(a)) \geq h_0 > 0 \). We can assume that \( m(E^-(a)) \geq h_0/2 \) (replacing \( f(x) \) by \( f(-x) \) if necessary). By our condition, we always have

\[ 2f(x) \leq f(-1) + f(1+2x) = f(1+2x). \]

Introducing a linear map \( T(x) := 1 + 2x \), we see that

\[ T(E^-(a)) \subseteq E(2a), \quad \text{and} \quad m(E(2a)) \geq m\left(T(E^-(a))\right) = 2 \cdot m(E^-(a)) \geq h_0 > 0. \]

Here the key observation is that from \( m(E(a)) \geq h_0 > 0 \) it follows \( m(E(2a)) \geq h_0 > 0 \). By iteration,

\[ m(E(2^k a)) := m([-1,1] \cap \{ f \geq 2^k a \}) \geq h_0 > 0, \quad \forall k = 1,2,\ldots. \]

Since \( 2^k a \rightarrow +\infty \) as \( k \rightarrow \infty \), and \( |f| < \infty \) a.e., we get a desired contradiction.

\#3. Show that

\[ H_n(x) := (-1)^n e^{x^2} \left( e^{-x^2} \right)^{(n)} \]

are polynomials of degree \( n \) (the Hermite polynomials) satisfying

\[ \int_{-\infty}^{\infty} e^{-x^2} H_k H_n \, dx = 0 \quad \text{for} \quad k \neq n. \]

Derive the equality

\[ F(t,x) := \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{2tx-t^2}. \]

**Proof.** It is easy to see that \( H'_n = 2x H_n - n H_{n+1} \), and by induction, \( H_n \) is a polynomial of degree \( n \) for every \( n \). Since \( H_k^{(n)} = 0 \) for \( n > k \), integrating by parts implies

\[ \int_{-\infty}^{\infty} e^{-x^2} H_k H_n \, dx = \int_{-\infty}^{\infty} H_k \cdot (-1)^n \left( e^{-x^2} \right)^{(n)} \, dx = \int_{-\infty}^{\infty} H_k^{(n)} e^{-x^2} \, dx = 0. \]

By symmetry, this equality also holds true for \( n < k \). Finally, using the Taylor expansion

\[ f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} h^n \quad \text{with} \quad f(x) := e^{-x^2}, \quad h := -t, \]

we get

\[ F(t,x) = e^{x^2} e^{-(x-t)^2} = e^{2tx-t^2}. \]

\#4. Let \( \{x_n\} \) be a sequence in a Hilbert space \( \mathcal{H} \) such that \( ||x_n|| \leq 1 \) for all \( n \), and for each \( y \in \mathcal{H} \), we have \( (x_n,y) \rightarrow 0 \) as \( n \rightarrow \infty \). Show that there is a subsequence \( \{x_{n_j}\} \) such that

\[ \frac{1}{k} \cdot (x_{n_1} + \cdots + x_{n_k}) \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \]

**Proof.** Take \( n_1 = 1 \), and then for \( j = 2,3,\ldots \), choose \( n_j \) such that

\[ ||(x_{n_i}, x_{n_j})|| \leq \frac{1}{j^2} \quad \text{for all} \quad i < j. \]

Then \( y_k := \frac{1}{k} \cdot (x_{n_1} + \cdots + x_{n_k}) \) satisfy

\[ ||y_k||^2 = (y_k, y_k) = \frac{1}{k^2} \sum_{i=1}^{k} ||x_{n_i}||^2 + \frac{2}{k^2} \sum_{1 \leq i < j \leq k} (x_{n_i}, x_{n_j}) \leq \frac{1}{k} + \frac{2}{k^2} \sum_{j=1}^{k} \frac{1}{j} \leq \frac{3}{k} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty. \]