Appendix A. Compactness in Metric Spaces.

In the textbook by *Walter Rudin*, Principles of Mathematical Analysis, 3rd edition, 1976, the compactness is defined by the following **Heine-Borel property**: A subset K in a metric space X is **compact**, if

$$K \subset \bigcup_{\alpha} G_\alpha \quad \text{with open} \quad G_\alpha \quad \Longrightarrow \quad K \subset \bigcup_{j=1}^{n} G_{\alpha_j} \quad \text{for some finite subfamily of} \quad \{G_\alpha\}.$$

Theorem A1. *The Heine-Borel property is equivalent to the following* **Weierstrass property:** *every infinite subset $E \subset K$ has a limit point in K.*

Proof. The Weierstrass property follows from the Heine-Borel property by Theorem 2.37 in the textbook. It suffices to show that if the Heine-Borel property fails, then the Weierstrass property fails as well. Therefore, suppose that

$$K \subset \bigcup_{\alpha} G_\alpha \quad \text{with open} \quad G_\alpha \quad \text{without a finite subcover.}$$

Since Q_α are open,

$$\forall p \in K, \quad \exists r(p) \in \left\{1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots\right\} \quad \text{such that} \quad p \in B_{r(p)}(p) \subset G_\alpha \quad \text{for some} \quad \alpha.$$

Pick $p_1 \in K$ with the maximal possible $r(p_1)$, and then by induction, for $k = 2, 3, \dots$,

$$p_k \in K \setminus \left(\bigcup_{j=1}^{k-1} B_{r(p_j)}(p_j) \right)$$

with the maximal possible $r(p_k)$. The subset $E := \{p_1, p_2, \dots, p_n, \dots\} \subset K$ is infinite, because otherwise $\left\{B_{r(p_j)}(p_j) \subset G_{\alpha_j}\right\}$ would be a finite subcover of K, and correspondingly, $\left\{G_{\alpha_j}\right\}$ would also be a finite subcover of K.

We claim that the set E has no limit point. Suppose otherwise: let $p_0 \in K$ be a limit point of E. Consider two possible cases.

(i) $p_0 \in B_{r(p_k)}(p_k)$ for some k. Since all these balls are open, we have $p_0 \in B_\varepsilon(p_0) \subset B_{r(p_k)}(p_k)$ for some $\varepsilon > 0$. By constructions, all the point p_j with $j \geq k+1$ lie outside of $B_{r(p_k)}(p_k)$, hence $B_\varepsilon(p_0)$ can only contain a finite number of point p_j , and by Theorem 2.20, p_0 cannot be a limit point of E.

(ii) $p_0 \notin B_{r(p_k)}(p_k)$ for all k. Once again by construction, we must have $0 < r(p_0) \leq r(p_k)$ for all k. Then $d(p_0, p_k) \geq r(p_k) \geq r(p_0) > 0$, i.e. $p_k \notin B_{r(p_0)}(p_0)$ for all k, and p_0 is not a limit point of E.

In any case, the Weierstrass property fails for the infinite set E, which completes the proof. □

This theorem can be re-formulated in the following form. In one direction, this statement is contained in Theorem 3.6(a) in the textbook.

Theorem A2. *A subset K of a metric space (X, d) is compact if and only if every sequence $\{p_n\} \subset K$ contains a convergent subsequence in K.*

We give one more convenient criterion of compactness.

Definition A3. A subset E of a metric space (X, d) is **totally bounded** if $\forall \varepsilon > 0$, there exists a finite subset $\{p_1, p_2, \ldots, p_n\} \subset E$ such that

$$E \subset \bigcup_{j=1}^{n} B_\varepsilon(p_j).$$

Theorem A4. *A subset K of a metric space (X, d) is compact if and only if it is (i) complete and (ii) totally bounded.*

Proof. Let K be a compact subset of X. The completeness of K is contained in Theorem 3.11(b) in the textbook. Alternatively, one can use the above Theorem A1 in the following way. Let $\{p_n\}$ be a Cauchy sequence in K. If the set $E := \{p_n\} \subset K$ is finite, then we obviously have $p = p_{n_1} = p_{n_2} = \cdots$ for some sequence of natural indices $n_1 < n_2 < \cdots$. In this case trivially $p_{n_j} \to p$ as $j \to \infty$. If the set E is infinite, then by Theorem A1 it has a limit point $p \in K$. By Theorem 3.2(d), the set E contains a convergent subsequence $p_{n_j} \to p \in K$ as $j \to \infty$, so that this property holds true in any case. Next, since $\{p_n\}$ is a Cauchy sequence, $\forall \varepsilon > 0$, $\exists N = N(\varepsilon) > 0$ such that $d(p_n, p_m) < \varepsilon$ for all $m, n \ge N$. Then

$$d(p_n, p) \le d(p_n, p_{n_j}) + d(p_{n_j}, p) < \varepsilon + d(p_{n_j}, p), \quad \forall n,\ n_j \ge N.$$

By taking limit as $j \to \infty$, we get $d(p_n, p) \le \varepsilon$, $\forall n \ge N$. This implies that $\{p_n\}$ converges to p, so that K is complete.

Compact sets K must be totally bounded, because otherwise we get infinite set $E := \{p_n\} \subset K$ with $d(p_j, p_k) \ge \varepsilon > 0$, $\forall j \ne k$. Then E' is empty, in contradiction to Theorem A1.

Now suppose that K is complete and totally bounded, and let E be an infinite subset of K. Starting from $E_0 := E$ and using total boundedness, we can define a decreasing sequence of infinite sets

$$E_n := E_{n-1} \cap B_{1/n}(p_n) \quad \text{for some distinct points} \quad p_n \in K, \quad n = 1, 2, \ldots.$$

By this construction, we have

$$d(p_m, p_n) < \frac{1}{n}, \quad \forall m > n.$$

This implies that $\{p_n\}$ is a Cauchy sequence. By completeness, $p_n \to p \in E'$, so that E' is nonempty. By Theorem A1, the subset K is compact in (X, d). □