Math 8602: REAL ANALYSIS. Spring 2016
Some problems for Midterm Exam #1 on Wednesday, February 24.

You will have 50 minutes (10:10 am–11:00 am) to work on 4 problems, 2 of which will be selected from the following list, one out of problems 1–4, and another (certainly!) problem 5.
One of "new" problem is based on the previous material, such as the Lebesque dominated convergence theorem, another one - on the material from Chapter 3.

No books and electronic devices. You can use class notes, including Appendices and Solutions to Homeworks.

#1. Let

$$f(t) := 1 - \sum_{n=1}^{\infty} c_n t^n \quad \text{be the Maclaurin series for} \quad (1-t)^{1/2}.$$

(a). Show that this series converges absolutely and uniformly on compact subsets of $(-1,1)$, as does the termwise differentiated series $-\sum\limits_{n=1}^{\infty} n c_n t^{n-1}$.
It is known from calculus that this implies $f'(t) = -\sum\limits_{n=1}^{\infty} n c_n t^{n-1}$.

(b). Show that $f(t) = -2(1-t)f'(t)$ and conclude that $(1-t)^{-1/2} f(t) = \text{const}$ on $(-1,1)$. Since $f(0) = 1,\ f(t) = 1-t)^{1/2}$.

#2. Let $f, f_1, f_2, \cdots$ be Lebesque integrable functions on $\mathbf{R}^n$, such that

$$\int |f_k - f| \to 0 \qquad \text{as} \quad k \to \infty. \tag{1}$$

Show that

(a)

$$\sup_k \int |f_k| \le C = const < \infty;$$

(b)

$$\sup_k \int_{\{|f_k| \ge N\}} |f_k| \to 0 \qquad \text{as} \quad N \to \infty.$$

#3 (#37, p. 108). Let F be a real function on $\mathbf{R}^1$. Show that F satisfies the Lipschitz condition

$$|F(x) - F(y)| \le M \cdot |x - y| \quad \text{forall} \quad x, y \in \mathbf{R}^1$$

with a constant $M \ge 0$ if and only if F is absolutely continuous and $|F'| \le M$ a.e. on $\mathbf{R}^1$.

#4. Show that the function

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(4^k x)}{2^k}$$

is continuous on $\mathbf{R}^1$ but its variation $V[f;\, a, b] = \infty$ for any $a < b$.

#5 (#13, p. 119). Let $(X, \mathcal{T})$ be a topological space, and let A be dense in X. Then for any open set U, we have $\overline{U} = \overline{U \cap A}$.