Math 8602. April 6, 2016. Midterm Exam 2. Problems and Solutions.

Problem 1. Show that for any two Borel measurable sets $E_1, E_2 \subseteq \mathbb{R}^1$ with finite Borel measure, the convolution

$$f(x) = \big(I_{E_1} * I_{E_2}\big)(x) := \int\limits_{\mathbb{R}^1} I_{E_1}(x-y) I_{E_2}(y)\, dy,$$

where $I_E(y) = 1$ if $y \in E$, and $I_E(y) = 0$ if $y \notin E$, is continuous on $\mathbb{R}^1$.

Proof. Both functions I_{E_1} and I_{E_2} belong to $L^1 := L^1(\mathbb{R}^1)$, so that they can be approximated in L^1 by continuous functions with compact support (Theorem 2.41 on p.70). For a fixed $\varepsilon > 0$, choose such a function g for which the norm in L^1, $||I_{E_1} - g||_1 \le \varepsilon/2$. Then the function $f_1 := g * I_{E_2}$ satisfies

$$\big|(f - f_1)(x)\big| = \Big|(I_{E_1} - g) * I_{E_2}(x)\Big| \le ||I_{E_1} - g||_1 \le \varepsilon/2, \quad \forall x \in \mathbb{R}^1.$$

Hence

$$|f(x+h) - f(x)| \le |f_1(x+h) - f_1(x)| + 2 \cdot \sup |f - f_1| \le |f_1(x+h) - f_1(x)| + \varepsilon, \quad \forall x, h \in \mathbb{R}^1.$$

By Theorem 2.27 (a) on p.56, the function f_1 is continuous on $\mathbb{R}^1$. Therefore,

$$\limsup_{h \to 0} |f(x+h) - f(x)| \le \varepsilon, \quad \forall x \in \mathbb{R}^1.$$

Since $\varepsilon > 0$ is arbitrary, f is continuous.

Problem 2. (a). Let K be a nonempty closed set in $\mathbb{R}^n$. Show that for every $\varepsilon > 0$, the sets

$$K^\varepsilon := \{x \in \mathbb{R}^n : \ \mathrm{dist}(x, K) \le \varepsilon\} \quad \text{are closed in} \quad \mathbb{R}^n.$$

(b). Let K_j be a sequence of nonempty compact sets in $\mathbb{R}^n$, such that $K_1 \supseteq K_2 \supseteq \cdots$. Set $K := \bigcap\limits_j K_j$. Show that for every $\varepsilon > 0$, there exists a constant $N = N(\varepsilon)$ such that $K_j \subseteq K^\varepsilon$, $\forall j \ge N$.

Proof. (a). Taking the infimum over $y \in K$ in the inequality $|x_1 - y| \le |x_2 - y| + |x_1 - x_2|$, we see that the distance function $d(x) := \mathrm{dist}(x, K) := \inf\{|x - y| : \ y \in K\}$ satisfies $d(x_1) \le d(x_2) + |x_1 - x_2|$. By symmetry, we always have

$$|d(x_1) - d(x_2)| \le |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n.$$

In particular, $d(x)$ is continuous, and $K^\varepsilon = d^{-1}([0, \varepsilon])$ is closed.

(b). The compact set

$$F := K_1 \cap \{d(x) \geq \varepsilon\} \subseteq K^c = \bigcup_{j=1}^{\infty} K_j^c,$$

where K_j^c are open. Therefore, $F \subseteq K_N^c$ for some N, and

$$K_j \subseteq K_N \subseteq (F^c) \cap K_1 \subseteq \{d(x) < \varepsilon\} \subseteq K^\varepsilon, \quad \forall j \geq N.$$

Problem 3. Let $f, f_1, f_2, \cdots$ be real measurable functions on $\mathbb{R}$, such that $f_n \to f$ almost everywhere (a.e.) as $n \to \infty$, and

$$\int_{\mathbb{R}} f(x)\, dx = 1, \quad \int_{\mathbb{R}} f_n(x)\, dx = 1, \quad \text{and} \quad f_n \geq 0 \quad \text{for all } n.$$

(a). Show that $f_n \to f$ in $L^1(R)$ as $n \to \infty$.

(b). Show that (a) may fail if the assumption $\int_{\mathbb{R}} f(x)\, dx = 1$ is dropped.

(c). Show that (a) may fail if the assumption $f_n \geq 0$ is dropped.

Proof. (a). We have $0 \leq g_n := \min\{f, f_n\} \leq f \in L^1$, and $g_n \to f$ a.e. By the Dominated Convergence Theorem,

$$\int_{\mathbb{R}} g_n dx \;\to\; \int_{\mathbb{R}} f\, dx = 1 \quad \text{as} \quad n \to \infty.$$

Moreover, it is easy to see that $|f_n - f| = f_n + f - 2g_n$. Therefore,

$$\int_{\mathbb{R}} |f_n - f|\, dx = \int_{\mathbb{R}} f_n dx + \int_{\mathbb{R}} f\, dx - 2\int_{\mathbb{R}} g_n dx = 2 - 2\int_{\mathbb{R}} g_n dx \;\to\; 0 \quad \text{as} \quad n \to \infty.$$

(b). $f_n := I_{(n,n+1)} \;\to\; f \equiv 0$ a.e., but not in L^1.

(c). $f_n := I_{(0,1)} - I_{(n,n+1)} + I_{(n+1,n+2)} \;\to\; f := I_{(0,1)}$ a.e., but not in L^1.

Problem 4. Let $f(x)$ be a continuous function on $[-1, 1]$, such that

$$\int_{-1}^{1} x^k f(x)\, dx = 0 \quad \text{for all} \quad k = 0, 1, 2, \ldots.$$

Show that $f \equiv 0$ on $[-1, 1]$.

Proof. From linearity it follows that

$$\int_{-1}^{1} pf\, dx = 0 \quad \text{for every polynomial} \quad p.$$

By the Weierstrass theorem, $f(x)$ can be approximated by a sequence of polynomials p_n uniformly on $[-1, 1]$. Then we must have

$$\int_{-1}^{1} f^2 dx = \lim_{n\to\infty} \int_{-1}^{1} p_n f\, dx = 0, \quad \text{and} \quad f \equiv 0 \quad \text{on} \quad [-1, 1].$$