Math 8602: REAL ANALYSIS. Spring 2016

Homework #4 (due on Wednesday, March 30).
40 points are divided between 4 problems, 10 points each.

#1. Let $\mathcal{K}$ be a family of all closed subset of $[0,1]\times[0,1]$ with respect to the Euclidean distance. Show that $\mathcal{K}$ is a metric space with the Hausdorff distance

$$\begin{aligned}\rho(A,B) &:= \max\Big\{\max_{x\in A}\operatorname{dist}(x,B),\ \max_{y\in B}\operatorname{dist}(y,A)\Big\},\\ \operatorname{dist}(x,B) &:= \min_{y\in B}|x-y|, \quad \text{etc.}\end{aligned}$$

#2. Show that in the previous problem, the metric space $(\mathcal{K},\rho)$ is compact.

#3 (Problem 63, p. 138). Let $K(x,y)$ be a continuous function on $[0,1]\times[0,1]$. Consider the metric space $\big(C([0,1]),\rho\big)$, where

$$\rho(f,g) := \max_{[0,1]}|f-g|.$$

Show that the family of functions

$$A := \left\{F(x) := \int_0^1 K(x,y)\,f(y)\,dy : \quad f\in C([0,1]), \quad \max_{[0,1]}|f|\le 1\right\}$$

is a precompact subset of $\big(C([0,1]),\rho\big)$. Verify whether or not it is compact.

#4 (§4.3, p. 147.) Let $(\mathcal{F},\lesssim)$ be a filter directed under reverse inclusion, i.e.

$$F_1\lesssim F_2 \quad\Longleftrightarrow\quad F_2\subseteq F_1.$$

A net $<x_F>_{F\in\mathcal{F}}$ is **associated** to $\mathcal{F}$ if $x_F\in F$ for every $F\in\mathcal{F}$. Show that

$$\mathcal{F}\ \to\ x \quad\Longleftrightarrow\quad \text{every associated net} \quad <x_F>_{F\in\mathcal{F}}\to\ x.$$