Math 8602: REAL ANALYSIS.   Spring 2016

Homework #1. Problems and Solutions.

#1. Let C be a collection of open balls in $\mathbb{R}^n$. Show that there exists a finite or countable subset $C_1 \subseteq C$ such that

$$\bigcup_{B \in C_1} B = \bigcup_{B \in C} B.$$

Proof. We can write

$$G := \bigcup_{B \in C} B = \bigcup_{j=1}^{\infty} K_j, \quad \text{where} \quad K_j := \left\{ x \in G : \quad |x| \le j, \quad \operatorname{dist}(x, \partial G) \ge \frac{1}{j} \right\}.$$

For each j, K_j is a compact, so that it can be covered by a finite subset of open balls $C_{1,k} \subseteq C$. Finally, the desired equality holds true with $C_1 := \bigcup_k C_{1,k}$.

#2. By definition on p. 95, a measurable function $f: \ \mathbb{R}^n \to \mathbb{R}$ is **locally integrable** ($f \in L^1_{loc}$) if

$$\int_K |f(x)|\, dx < \infty \qquad \forall \quad \text{bounded measurable} \quad K \subset \mathbb{R}^n.$$

Show that this definition is equivalent to the following:

$$\forall x \in \mathbb{R}^n, \quad \exists r > 0 \quad \text{such that} \quad \int_{B_r(x)} |f(y)|\, dy < \infty.$$

Proof. It suffices to show that from the second definition it follows the first one. Replacing K by its closure, we can assume that it is a compact. Then one can choose a finite set of balls $B_{r_j}(x_j),\ j = 1, 2, \dots, N$, from the second definition, which covers K. Then

$$\int_K |f(x)|\, dx \le \sum_{j=1}^{\infty} \int_{B_{r_j}(x_j)} |f(x)|\, dx < \infty.$$

#3. Let $d\nu = d\lambda + f\, dm$ be the Lebesgue-Radon-Nikodym decomposition of a finite real signed measure on $\mathbb{R}^n$. Show that for the total variations (defined on p. 87) we also have

$$d|\nu| = d|\lambda| + |f|\, dm.$$

Proof. Since $\lambda \perp \mu$, where $d\mu = f\, dm$, there are sets $E, F \in \mathcal{B} = \mathcal{B}(\mathbb{R}^n)$ such that

$$E \cap F = \emptyset, \quad E \cup F = \mathbb{R}^n, \quad E \quad \text{is null for} \quad \lambda, \quad \text{and} \quad F \quad \text{is null for} \quad \mu.$$

Let $\mathbb{R}^n = P_1 \cup N_1 = P_2 \cup N_2$ be Hahn decompositions for λ and μ respectively. Then

$$\mathbb{R}^n = P \cup N, \quad \text{where} \quad P := (P_1 F) \cup (P_2 E), \quad N := (N_1 F) \cup (N_2 E)$$

is a Hahn decompositions for λ, μ, and $\nu = \lambda + \mu$. As in the proof on the Jordan Decomposition Theorem 3.4 and definition of total variation $|\nu|$ on p.87, we have for all $E \in \mathcal{B}$:

$$\begin{aligned} |\nu(E)| &= \nu^+(E) + \nu^-(E) = \nu(EP) - \nu(EN), \\ |\lambda(E)| &= \lambda^+(E) + \lambda^-(E) = \lambda(EP) - \lambda(EN), \\ |\mu(E)| &= \mu^+(E) + \mu^-(E) = \mu(EP) - \mu(EN), \end{aligned}$$

which implies $|\nu| = |\lambda| + |\mu|$. Finally, since $d\mu = f\,dm$, we must have $f \ge 0$ a.e. on P, and $f \le 0$ a.e. on N, which gives $d|\mu| = |f|\,dm$.

#4. For each $x \in [0,1]$, let

$$x = \sum_{k=1}^{\infty} \frac{x_k}{2^k},$$

where $x_k = 0$ or 1, so that x_k are functions of x with values 0 and 1. Show that

$$S_n(x) = \frac{x_1 + x_2 + \cdots + x_n}{n} \to \frac{1}{2} \qquad \text{as} \quad n \to \infty \quad \text{in measure on} \quad [0,1].$$

Proof. Denote

$$I_{j,m} := \big(2^{-m}(j-1), 2^{-m}j\big) \quad \text{for} \quad m = 1, 2, \dots;\ j = 1, 2, \dots, 2^m.$$

We have

$$x_m(x) = \begin{cases} 0 & \text{if} \quad x \in I_{j,m} \quad \text{with an odd} \quad j; \\ 1 & \text{if} \quad x \in I_{j,m} \quad \text{with an even} \quad j. \end{cases}$$

For natural $n > m$, each interval $I_{j,m}$ is represented as a union of 2^{n-m} subintervals $I_{k,n}$, plus a finite number of their endpoints. On the interval $I_{j,m}$, the function $f_n(x) = x_n(x) - \frac{1}{2}$ alternates between $-\frac{1}{2}$ and $\frac{1}{2}$ and has zero integral, while $f_m(x) = \text{const}$ (which is either 0 or 1). Hence for $n > m$,

$$\int_0^1 f_m f_n \, dx = \sum_{j=1}^{2^m} \int_{\Delta_{j,m}} f_m f_n \, dx = 0.$$

By symmetry, the functions $f_k(x)$ are orthogonal in $L^2([0,1])$. Further, note that

$$S_n(x) - \frac{1}{2} = \frac{1}{n} \sum_{m=1}^{n} f_m(x).$$

Applying Chebyshev's inequality, we how have for any $\alpha > 0$:

$$m\big(\{x:\ |S_n(x) - 1/2| > \alpha\}\big) = m\big(\{x:\ |S_n(x) - 1/2|^2 > \alpha^2\}\big)$$

$$\le \frac{1}{\alpha^2} \int_0^1 |S_n(x) - 1/2|^2 dx = \frac{1}{n^2\alpha^2} \int_0^1 \left(\sum_{m=1}^{n} f_m \right)^2 dx = \frac{1}{n^2\alpha^2} \int_0^1 \sum_{k,m=1}^{n} f_k f_m \, dx$$

$$= \frac{1}{n^2\alpha^2} \int_0^1 \sum_{m=1}^{n} f_m^2 \, dx = \frac{1}{n^2\alpha^2} \cdot \frac{n}{2} = \frac{1}{4n\alpha^2} \to 0 \quad \text{as} \quad n \to \infty.$$

This means that $S_n(x) \to \frac{1}{2}$ in measure.