New Exotic 4-manifolds via Luttinger surgery on Lefschetz fibrations

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(A joint work with Prof. Anar Akhmedov)

Differential Geometry and Symplectic Topology Seminar

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2 Background
3 Symplectic Building Blocks
4 Construction of the exotic \((2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}^2}\)
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Let $M$ be $(2n + 2k - 3)CP^2 \# (6n + 2k - 3)\overline{CP}^2$ for any $n \geq 1$, $k \geq 1$, and $(n, k) \neq (1, 1)$. There exists a new family of smooth, closed, simply-conn., minimal, symp. 4-man. and an infinite family of non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to $M$ that can obtained by the sequence of Luttinger surgeries and a single generalized torus surgery on Lefschetz fibrations.
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(2) Lefschetz Fibration
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Definition

Let $\Sigma_g$ denote a 2-dim., closed, oriented, and connected surface of genus $g > 0$.

Let $\text{Diff}^+ (\Sigma_g)$ denote the group of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$ and $\text{Diff}_0^+ (\Sigma_g)$ be the subgroup of $\text{Diff}^+ (\Sigma_g)$ consisting of all orientation-preserving diffeomorphisms $\Sigma_g \to \Sigma_g$ that are isotopic to the identity.

The mapping class group $M_g$ of $\Sigma_g$ is defined to be the group of isotopy classes of orientation-preserving diffeomorphisms of $\Sigma_g$, i.e.,

$$M_g = \text{Diff}^+ (\Sigma_g) / \text{Diff}_0^+ (\Sigma_g).$$

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Let $\alpha$ be a simple closed curve on $\Sigma_g$.

A right handed Dehn twist $t_\alpha$ about $\alpha$ is the isotopy class of a self-diffeomorphism of $\Sigma_g$ obtained by cutting the surface $\Sigma_g$ along $\alpha$ and gluing the ends back after rotating one of the ends $2\pi$ to the right.
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Definition: Lefschetz Fibration

Let $X$ be a compact, connected, oriented, smooth 4-manifold. A [Lefschetz fibration on $X$](https://en.wikipedia.org/wiki/Lefschetz_fibration) is a smooth map $f : X \rightarrow \Sigma_h$, such that

1. $f$ is surjective
2. Each critical point of $f$ has an orientation preserving chart on which $f : C^2 \rightarrow \Sigma_h$ is given by $f(z_1, z_2) = z_1^3 + z_2^2$.

Remark

$f$ is a smooth fiber bundle away from finitely many critical points. Let us denote the critical points of $f$ by $p_1, \ldots, p_r$. 

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The genus of the regular fiber of $f$ is defined to be \textit{the genus of the L. F.}.

\textbf{singular fiber}

A fiber of $f$ passing through the critical point set $p_1, \ldots, p_s$ is called a \textit{singular fiber}, which is an immersed surface with a single transverse self-intersection. A singular fiber of the genus $g$ Lefschetz Fibration can be described by its monodromy, i.e., an element of the mapping class group $M_g$.

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This element is a right-handed (or a positive) Dehn twist along a simple closed curve on $\Sigma_g$, called the \textit{vanishing cycle}. If this curve is a nonseparating curve, then the singular fiber is called \textit{nonseparating}, otherwise it is called \textit{separating}.
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L.F. $\leftrightarrow$ Global Monodromy

For a genus $g$ Lefschetz fibration over $S^2$, the product of right handed Dehn twists $t_{\alpha_i}$ along the vanishing cycles $\alpha_i$, for $i = 1, \ldots, s$, determines the global monodromy of the Lefschetz fibration, the relation $t_{\alpha_1} \cdot t_{\alpha_2} \cdots t_{\alpha_s} = 1$ in $M_g$. Conversely, such a relation in $M_g$ determines a genus $g$ Lefschetz fibration over $S^2$ with the vanishing cycles $\alpha_1, \ldots, \alpha_s$.

Lemma

Let $f : X \to S^2$ be a genus $g$ Lefschetz fibration with global monodromy given by the relation $t_{\alpha_1} \cdot t_{\alpha_2} \cdots t_{\alpha_s} = 1$.

Suppose that $f$ has a section.

Then we have $\pi_1(X) \cong \pi_1(\Sigma_g)/\langle \alpha_1, \alpha_2, \cdots, \alpha_s \rangle$.

In particular, there is an epimorphism $\pi_1(\Sigma_g) \to \pi_1(X)$.
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Example (Genus one Lefschetz fibrations)

Let $M_1$ be the mapping class group of $\mathbb{T}^2 = a \times b$. $M_1 = SL(2, \mathbb{Z})$ is generated by

$$t_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$t_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

with relations $t_at_bt_a = t_bt_at_b$ and $(t_at_b)^6 = 1$.

The monodromy relation given by $(t_at_b)^6n = 1$, corresponding to the genus 1 Lefschetz fibration over $S^2$, has total space $E(n)$.

$E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}}^2$

$E(2)$, $K3$ surface.
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Let $M_1$ be the mapping class group of $\mathbb{T}^2 = a \times b$. 
$M_1 = SL(2, \mathbb{Z})$ is generated by

$$t_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
$$t_b = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

with relations $t_a t_b t_a = t_b t_a t_b$ and $(t_a t_b)^6 = 1$.

The monodromy relation given by $(t_a t_b)^6 = 1$, corresponding to the genus 1 Lefschetz fibration over $S^2$, has total space $E(n)$.

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Example (Hyperelliptic Lefschetz fibrations)

Let $\alpha_1, \alpha_2, \ldots, \alpha_{2g}, \alpha_{2g+1}$ denote the collection of simple closed curves given in the following figure, and $c_i$ denote the right handed Dehn twists along the curve $\alpha_i$. The following relations hold in the mapping class group $M_g$:

$$\Gamma_1(g) = (c_1 c_2 \cdots c_{2g-1} c_{2g}^2 c_{2g+1} c_{2g} c_{2g-1} \cdots c_2 c_1)^2 = 1.$$  
$$\Gamma_2(g) = (c_1 c_2 \cdots c_{2g-1} c_{2g} c_{2g+1})^{2g+2} = 1.$$  
$$\Gamma_3(g) = (c_1 \cdot c_2 \cdots c_{2g-1} c_{2g})^{2(2g+1)} = 1.$$  

The monodromy relation given by $\Gamma_1(g)$, corresponding to the genus $g$ Lefschetz fibration over $S^2$, has total space $X(g, 1) = \mathbb{CP}^2 \# (4g + 5)\overline{\mathbb{CP}}^2$. Furthermore, for $g \geq 2$, the above fibration on $X(g, 1)$ admits $4g + 4$ disjoint $-1$-sphere sections.
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Vanishing Cycles of the Genus $g$ Lefschetz Fibration on $X(g, 1) = \mathbb{CP}^2 \# (4g + 5)\overline{\mathbb{CP}}^2$
Definition: Symplectic Connected Sum

Let $X_1$ and $X_2$ be closed, oriented, smooth 4-manifolds each containing a smoothly embedded surface $F_i$ of genus $g \geq 1$. Assume $F_i$ represents a homology of infinite order with $[F_1]^2 + [F_2]^2 = 0$, so that there exist a tubular neighborhood, say $\nu F_i \cong F_i \times D^2$, in $X_i$. Using an orientation-reversing diffeomorphism $\psi: S^1 \times F_1 \to S^1 \times F_2$, we can glue $X_1 \setminus \nu F_1$ and $X_2 \setminus \nu F_2$ along the boundary $\partial(\nu F_i) \cong F_i \times S^1$. This new oriented smooth 4-man $X_1 \#_\psi X_2$ is called a \textit{symplectic connected sum} of $X_1$ and $X_2$ along $F_i$. 
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Lemma

Let $X$ and $Y$ be closed, oriented, smooth 4-manifolds containing an embedded genus $g$ surface $\Sigma$ with $\Sigma^2 = 0$. Then

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\begin{align*}
c_1^2(X \#_\psi Y) &= c_1^2(X) + c_1^2(Y) + 8(g - 1), \\
\chi_h(X \#_\psi Y) &= \chi_h(X) + \chi_h(Y) + (g - 1).
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Proof

The above formulas simply follow from the well-known formulas

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\begin{align*}
e(X \#_\psi Y) &= e(X) + e(Y) - 2e(\Sigma), \\
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once we apply the formulas $c_1^2 = 3\sigma + 2e$ and $\chi_h = (\sigma + e)/4$. 
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**Remark**

If $X$, $Y$ are symplectic manifolds and $\Sigma$ is a symplectic submanifold then according to theorem of Gompf $X \# \psi Y$ admits a symplectic structure.

**Theorem, Usher: Minimality of Symplectic Sums**

Let $Z = X_1 \#_{F_1} F_2 X_2$ be symplectic fiber sum of manifolds $X_1$ and $X_2$. Then:

(i) If either $X_1 \setminus F_1$ or $X_2 \setminus F_2$ contains an embedded symplectic sphere of square $-1$, then $Z$ is not minimal.

(ii) If one of the summands $X_i$ (say $X_1$) admits the structure of an $S^2$-bundle over a surface of genus $g$ such that $F_i$ is a section of this fiber bundle, then $Z$ is minimal if and only if $X_3$ is minimal.

(iii) In all other cases, $Z$ is minimal.
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Definition: Luttinger Surgery

Let \((X, \omega)\) be a symplectic 4-manifold, and the torus \(\Lambda\) be a Lagrangian submanifold of \(X\) with \(\Lambda^2 = 0\). Given a simple loop \(\lambda\) on \(\Lambda\), let \(\lambda'\) be a simple loop on \(\partial(\nu \Lambda)\) that is parallel to \(\lambda\) under the Lagrangian framing.

For any integer \(m\), The \((\Lambda, \lambda, 1/m)\) Luttinger surgery on \(X\) will be

\[
X_{\Lambda, \lambda}(1/m) = (X - \nu(\Lambda)) \cup_\phi (S^1 \times S^1 \times D^2)
\]

Here \(\phi : S^1 \times S^1 \times \partial D^2 \to \partial(X - \nu(\Lambda))\) denotes a gluing map satisfying \(\phi([\partial D^2]) = m[\lambda'] + [\mu_{\Lambda}]\) in \(H_1(\partial(X - \nu(\Lambda)))\), where \(\mu_{\Lambda}\) is a meridian of \(\Lambda\).

Theorem, [D. Auroux, S.K. Donaldson, L. Katzarkov]

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For any integer \(m\), The \((\Lambda, \lambda, 1/m)\) Luttinger surgery on \(X\) will be

\[
X_{\Lambda, \lambda}(1/m) = (X - \nu(\Lambda)) \cup_{\phi} (S^1 \times S^1 \times D^2)
\]

Here \(\phi : S^1 \times S^1 \times \partial D^2 \rightarrow \partial (X - \nu(\Lambda))\) denotes a gluing map satisfying \(\phi([\partial D^2]) = m[\lambda'] + [\mu_\Lambda]\) in \(H_1(\partial (X - \nu(\Lambda)))\), where \(\mu_\Lambda\) is a meridian of \(\Lambda\).

Theorem, [D. Auroux, S.K. Donaldson, L. Katzarkov]

\(X_{\Lambda, \lambda}(1/m)\) possesses a symplectic form that restricts \(\omega\) on \(X \setminus \nu \Lambda\).
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(1) \( b_1(M_\Lambda, \lambda(1/m)) = b_1(X) - 1 \)
\( b_2(M_\Lambda, \lambda(1/m)) = b_2(X) - 2 \)

(2) \( e(M_\Lambda, \lambda(1/m)) = e(X) \)

(3) \( \sigma(M_\Lambda, \lambda(1/m)) = \sigma(M) \)

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Building Blocks
(1) Korkmaz's fibration
(2) Gurtas' s fibration
(3) Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$
(4) Luttinger surgeries on product manifolds $\Sigma_n \times \mathbb{T}^2$
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Y. Matsumoto: \( Y(1) = T^2 \times S^2 \# 4\mathbb{CP}^2 \)

M. Korkmaz: \( k \geq 2 \), by factorizing the vertical involution of \( \theta \) of genus \( 2k \) surface. \( Y(1, k) = \Sigma_k \times S^2 \# 4\mathbb{CP}^2 \).

Y. Gurtas: \( Y(n, k) = \Sigma_k \times S^2 \# 4n\mathbb{CP}^2 \), \( k, n \geq 1 \) \( (k, n) \neq (1, 1) \)
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The vertical involution $\theta$ of the genus $2k$ surface
The branch locus for $\Sigma_k \times S^2 \# 4\overline{CP}^2$

\[ \Sigma_k \times \{pt\} \quad \Sigma_k \times \{pt\} \]

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The branched-cover description of Korkmaz’ s fibration

Take a double branched cover of $\Sigma_k \times S^2$ along the union of 2 disjoint copies of $\Sigma_k \times \{pt\}$ and two disjoint copies of $\{pt\} \times S^2$.

The resulting branched cover has 4 singular points, corresponding to the number of the intersections points of the vertical genus $k$ surfaces and the horizontal spheres in the branch set.

Desingularize this singular manifold to obtain $Y(k) = \Sigma_k \times S^2 \# 4\mathbb{CP}^2$. 
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Theorem, M. Korkmaz

Let $\theta$ denote the vertical involution of the genus $g$ surface with 2 fixed points. In the mapping class group $M_g$, the following relations between right Dehn twists hold:

a) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g})^2 = \theta^2 = 1$ if $g$ is even,

b) $(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} (t_a)^2 (t_b)^2)^2 = \theta^2 = 1$ if $g$ is odd,

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The vanishing cycles
Let $\Sigma_{2k}$ denote a regular fiber of the given Lefschetz fibration and $a_1, b_1, \ldots, a_{2k}$ and $b_{2k}$ denote the standard generators of fundamental group of $\Sigma_{2k}$ under the inclusion.

Using the homotopy exact sequence for a Lefschetz fibration, we have $\pi_1(\Sigma_{2k}) \to \pi_1(Y(k)) \to \pi_1(S^2)$.

According to M. Korkmaz, we have the following identification of the fundamental group of $Y(k)$: $\pi_1(Y(k)) = \pi_1(\Sigma_{2k}) \setminus \langle B_0, B_1, \cdots, B_{g-1}, B_g, c \rangle$.

It follows that the fundamental group of $Y(k)$ has a presentation with the generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$ and the relations $[a_1, b_1][a_2, b_2] \cdots [a_g, b_g] = 1, B_0 = B_1 = B_2 = \cdots = B_g = c = 1$. 

Nur Saglam (University of Minnesota, School of Mathematics)
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$\pi_1(Y(k))$

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\[ \pi_1(Y(k)) \]

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\( \pi_1(Y(k)) \)

**Lemma**

The following relations hold in \( \pi_1(Y(k)) \)

\[
\begin{align*}
    a_1a_{2k} &= 1, \quad a_2a_{2k-1} = 1, \ldots, \quad a_ka_{k+1} = 1, \\
    b_1b_2 \cdots b_{2k} &= 1, \\
    b_2b_3 \cdots b_{2k-1} &= [a_{2k}, b_{2k}], \\
    \ldots \\
    b_{i+1}b_{i+2} \cdots b_{g-i} &= [a_{2k-i+1}, b_{2k-i+1}] \cdots [a_{2k-1}, b_{2k-1}][a_{2k}, b_{2k}]
\end{align*}
\]

**Conclusion**

The fundamental group of \( Y(k) = \Sigma \times \mathbb{S}^2 \# 4\mathbb{CP}^2 \) is isomorphic to the surface group \( \prod \pi_1(\Sigma \# 4\mathbb{CP}^2) \), generated by loops \( a_1, b_1, \ldots, a_k \) and \( b_k \).

Furthermore, the fundamental group of the complement of \( Y(k) \setminus \nu(\Sigma \# 4\mathbb{CP}^2) \) is also \( \prod \pi_1 \).

The normal circle \( \mu = \{pt\} \times \partial(D^2) \) to \( \Sigma \# 4\mathbb{CP}^2 \) can be deformed using an exceptional sphere section, thus trivial in \( \pi_1(Y(k) \setminus \nu(\Sigma \# 4\mathbb{CP}^2)) \).
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The following relations hold in $\pi_1(Y(k))$

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\begin{align*}
    a_1 a_2 &= 1, \quad a_2 a_{2k-1} = 1, \ldots, \quad a_k a_{k+1} = 1, \\
    b_1 b_2 \cdots b_{2k} &= 1, \\
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Conclusion

The fundamental group of $Y(k) = \Sigma_k \times S^2 \# 4\mathbb{CP}^2$ is isomorphic to the surface group $\prod_k = \pi_1(\Sigma_k)$, generated by loops $a_1, b_1, \ldots, a_k$ and $b_k$.

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Yusuf Gurtas generalized the constructions even further. He presented the positive Dehn twist expression for a new set of involutions in the mapping class group $\mathcal{M}_{h+v}$ of a closed, oriented 2-dimensional surface $\Sigma_{h+v}$.

These involutions were obtained by gluing the horizontal involution on a surface $\Sigma_h$ and the vertical involution on a surface $\Sigma_v$, where $v$ is a positive even number.

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The involution $\theta$ of the surface $\Sigma_{2k+n-1}$
We set \( h = n - 1 \) and \( v = 2k \).

Let \( Y(n, k) \) denote the total space of the Lefschetz fibration defined by the word \( \theta^2 = 1 \) in the mapping class group \( M_{2k+n-1} \).

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s = 8h + 2v + 4 = 8(n - 1) + 2(2k) + 4 = 8n + 4k - 4\]

singular fibers and the vanishing cycles all are about nonseparating curves.

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c_1^2(Y(n, k)) = -4(n + 2k - 2) \quad \text{and} \quad \chi_h(Y(n, k)) = 1 - k
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The branch locus for $\Sigma_k \times S^2 \# 4n\mathbb{CP}^2$
The branched cover description of Gurtas’ s fibrations

Take a double branched cover of $\Sigma_k \times S^2$ along the union of $2n$ disjoint copies of $\{pt\} \times S^2$ and two disjoint copies of $\Sigma_k \times \{pt\}$.

The resulting branched cover has $4n$ singular points, corresponding to the number of the intersections points of the horizontal spheres and the vertical genus $k$ surfaces in the branch set.

We desingularize this manifold to obtain $Y(n,k) = \Sigma_k \times S^2 \# 4n\overline{\mathbb{CP}}^2$. 

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A generic horizontal fiber

A generic horizontal fiber is the double cover of $S^2$ branched over two pts. Thus, we have a sphere fibration on $Y(n, k) = \Sigma_k \times S^2 \# 4\overline{CP}^2$.

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A generic vertical fiber is the double cover of $\Sigma_k$ branched over $2n$ points. Thus, a generic fiber of the vertical fibration has genus $2k + n - 1$. Furthermore, two complicated singular fibers of the vertical fibration can be perturbed into $4n + 2k - 2$ Lefschetz type singular fibers.
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Theorem

The positive Dehn twist expression for the involution $\theta$ is given by

$$\theta = e_{2i+2} \cdots e_{2n-2}e_{2n-1}e_{2i} \cdots e_2e_1B_0e_{2n-1}e_{2n-2} \cdots e_{2i+2}e_1e_2 \cdots e_2B_1B_2 \cdots B_{4k-1}B_{4k}e_{2i+1}$$

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The genus $2k + n - 1$ Lefschetz fibration on $Y(n, k)$ admits at least $4n$ disjoint $-1$ sphere sections.

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Let us fix integers $n \geq 2$, $p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. We denote by $Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)$ the symplectic 4-manifold obtained by performing the following $2n + 4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$, which consist of the following 8 surgeries:

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Luttinger surgeries on product manifolds \( \Sigma_n \times \Sigma_2 \)

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\[(a'_1 \times c'_1, d'_1, -1), \quad (b'_1 \times c''_1, b'_1, -1), \quad (a'_2 \times c'_1, c'_1, +1/p_1), \quad (a''_2 \times d'_1, d'_1, +1/q_1), \]
\[(a'_2 \times c'_2, d'_2, -1), \quad (b'_2 \times c''_2, b'_2, -1), \quad (a'_1 \times c'_2, c'_2, +1/p_2), \quad (a''_1 \times d'_2, d'_2, +1/q_2), \]

followed by the following set of additional \( 2(n - 2) \) Luttinger surgeries
Let us fix integers \( n \geq 2, p_i \geq 0 \) and \( q_i \geq 0 \), where \( 1 \leq i \leq n \).
We denote by \( Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \) the symplectic 4-manifold obtained by performing the following \( 2n + 4 \) Luttinger surgeries on \( \Sigma_n \times \Sigma_2 \),

which consist of the following 8 surgeries:

\[
(a'_1 \times c'_1, a'_1, -1), \quad (b'_1 \times c''_1, b'_1, -1), \quad (a'_2 \times c'_1, c'_1, +1/p_1), \quad (a''_2 \times d'_1, d'_1, +1/q_1),
\]
\[
(a'_2 \times c'_2, a'_2, -1), \quad (b'_2 \times c''_2, b'_2, -1), \quad (a'_1 \times c'_2, c'_2, +1/p_2), \quad (a''_1 \times d'_2, d'_2, +1/q_2),
\]

followed by the following set of additional \( 2(n - 2) \) Luttinger surgeries.
Let us fix integers $n \geq 2$, $p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. We denote by $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$ the symplectic 4-manifold obtained by performing the following $2n + 4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$, which consist of the following 8 surgeries:

$$(a'_1 \times c'_1, a'_1, -1), \quad (b'_1 \times c''_1, b'_1, -1), \quad (a'_2 \times c'_1, c'_1, +1/p_1), \quad (a''_2 \times d'_1, d'_1, +1/q_1),$$

$$(a'_2 \times c'_2, a'_2, -1), \quad (b'_2 \times c''_2, b'_2, -1), \quad (a'_1 \times c'_2, c'_2, +1/p_2), \quad (a''_1 \times d'_2, d'_2, +1/q_2),$$

followed by the following set of additional $2(n - 2)$ Luttinger surgeries.
Let us fix integers $n \geq 2$, $p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. We denote by $Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)$ the symplectic 4-manifold obtained by performing the following $2n + 4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$, which consist of the following 8 surgeries:

$$(a'_1 \times c'_1, a'_1, -1), \quad (b'_1 \times c''_1, b'_1, -1), \quad (a'_2 \times c'_1, c'_1, +1/p_1), \quad (a''_2 \times d'_1, d'_1, +1/q_1),$$
$$(a'_2 \times c'_2, a'_2, -1), \quad (b'_2 \times c''_2, b'_2, -1), \quad (a'_1 \times c'_2, c'_2, +1/p_2), \quad (a''_1 \times d'_2, d'_2, +1/q_2),$$

followed by the following set of additional $2(n-2)$ Luttinger surgeries...

...
Let us fix integers \( n \geq 2, p_i \geq 0 \) and \( q_i \geq 0 \), where \( 1 \leq i \leq n \). We denote by \( Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \) the symplectic 4-manifold obtained by performing the following \( 2n + 4 \) Luttinger surgeries on \( \Sigma_n \times \Sigma_2 \),

which consist of the following 8 surgeries:

\[
\begin{align*}
(a_1' \times c_1', a_1', -1), & \quad (b_1' \times c_1'', b_1', -1), & \quad (a_2' \times c_1', c_1', +1/p_1), & \quad (a_2'' \times d_1', d_1', +1/q_1), \\
(a_2' \times c_2', a_2', -1), & \quad (b_2' \times c_2'', b_2', -1), & \quad (a_1' \times c_2', c_2', +1/p_2), & \quad (a_1'' \times d_2', d_2', +1/q_2),
\end{align*}
\]

followed by the following set of additional \( 2(n - 2) \) Luttinger surgeries
Let us fix integers $n \geq 2$, $p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. We denote by $Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)$ the symplectic 4-manifold obtained by performing the following $2n + 4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$, which consist of the following 8 surgeries:

$$
(a'_1 \times c'_1, a'_1, -1), \quad (b'_1 \times c''_1, b'_1, -1), \quad (a'_2 \times c'_1, c'_1, +1/p_1), \quad (a''_2 \times d'_1, d'_1, +1/q_1), \\
(a'_2 \times c'_2, a'_2, -1), \quad (b'_2 \times c''_2, b'_2, -1), \quad (a'_1 \times c'_2, c'_2, +1/p_2), \quad (a''_1 \times d'_2, d'_2, +1/q_2),
$$

followed by the following set of additional $2(n - 2)$ Luttinger surgeries:

$$
(b'_1 \times c''_1, b'_1, -1), \quad (a'_1 \times c'_2, c'_2, +1/p_2), \quad (a''_2 \times d'_1, d'_1, +1/q_1), \quad (a''_1 \times d'_2, d'_2, +1/q_2),
$$

$$
(a'_1 \times c'_1, a'_1, -1), \quad (b'_1 \times c''_1, b'_1, -1), \quad (a'_2 \times c'_1, c'_1, +1/p_1), \quad (a''_2 \times d'_1, d'_1, +1/q_1),
$$

$$
(a'_2 \times c'_2, a'_2, -1), \quad (b'_2 \times c''_2, b'_2, -1), \quad (a'_1 \times c'_2, c'_2, +1/p_2), \quad (a''_1 \times d'_2, d'_2, +1/q_2),
$$
Let us fix integers \( n \geq 2, p_i \geq 0 \) and \( q_i \geq 0 \), where \( 1 \leq i \leq n \).

We denote by \( Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \) the symplectic 4-manifold obtained by performing the following \( 2n + 4 \) Luttinger surgeries on \( \Sigma_n \times \Sigma_2 \),

which consist of the following 8 surgeries:

\[
(a'_1 \times c'_1, a'_1, -1), \quad (b'_1 \times c''_1, b'_1, -1), \quad (a'_2 \times c'_1, c'_1, +1/p_1), \quad (a''_2 \times d'_1, d'_1, +1/q_1), \\
(a'_2 \times c'_2, a'_2, -1), \quad (b'_2 \times c''_2, b'_2, -1), \quad (a'_1 \times c'_2, c'_2, +1/p_2), \quad (a''_1 \times d'_2, d'_2, +1/q_2),
\]

followed by the following set of additional \( 2(n - 2) \) Luttinger surgeries:

\[
(b'_1 \times c'_3, c'_3, -1/p_3), \quad (b'_2 \times d'_3, d'_3, -1/q_3), \\
\vdots \\
(b'_1 \times c'_n, c'_n, -1/p_n), \quad (b'_2 \times d'_n, d'_n, -1/q_n)
\]
Let us fix integers \( n \geq 2, p_i \geq 0 \) and \( q_i \geq 0 \), where \( 1 \leq i \leq n \).

We denote by \( \Sigma_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \) the symplectic 4-manifold obtained by performing the following \( 2n + 4 \) Luttinger surgeries on \( \Sigma_n \times \Sigma_2 \),

which consist of the following 8 surgeries:

\[
\begin{align*}
(a'_1 \times c'_1, a'_1, -1), & \quad (b'_1 \times c''_1, b'_1, -1), & \quad (a'_2 \times c'_1, c'_1, +1/p_1), & \quad (a''_2 \times d'_1, d'_1, +1/q_1), \\
(a'_2 \times c'_2, a'_2, -1), & \quad (b'_2 \times c''_2, b'_2, -1), & \quad (a'_1 \times c'_2, c'_2, +1/p_2), & \quad (a''_1 \times d'_2, d'_2, +1/q_2),
\end{align*}
\]

followed by the following set of additional \( 2(n - 2) \) Luttinger surgeries:

\[
\begin{align*}
(b'_1 \times c'_3, c'_3, -1/p_3), & \quad (b'_2 \times d'_3, d'_3, -1/q_3), \\
\cdots & \quad \cdots \\
(b'_1 \times c'_n, c'_n, -1/p_n), & \quad (b'_2 \times d'_n, d'_n, -1/q_n)
\end{align*}
\]
Let us fix integers $n \geq 2$, $p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. We denote by $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$ the symplectic 4-manifold obtained by performing the following $2n + 4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$, which consist of the following 8 surgeries:

$$(a_1' \times c_1', a_1', -1), \quad (b_1' \times c''_1, b_1', -1), \quad (a_2' \times c_1', c_1', +1/p_1), \quad (a_2'' \times d_1', d_1', +1/q_1),$$
$$(a_2' \times c_2', a_2', -1), \quad (b_2' \times c''_2, b_2', -1), \quad (a_1' \times c_2', c_2', +1/p_2), \quad (a_1'' \times d_2', d_2', +1/q_2),$$

followed by the following set of additional $2(n - 2)$ Luttinger surgeries:

$$(b_1' \times c_3', c_3', -1/p_3), \quad (b_1' \times c_n', c_n', -1/p_n),$$
$$(b_2' \times d_3', d_3', -1/q_3),$$
$$\cdots$$
$$(b_2' \times d_n', d_n', -1/q_n)$$
Let us fix integers $n \geq 2$, $p_i \geq 0$ and $q_i \geq 0$, where $1 \leq i \leq n$. We denote by $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$ the symplectic 4-manifold obtained by performing the following $2n + 4$ Luttinger surgeries on $\Sigma_n \times \Sigma_2$, which consist of the following 8 surgeries:

$$(a_1' \times c_1', a_1', -1), \quad (b_1' \times c_1'', b_1', -1), \quad (a_2' \times c_1', c_1', +1/p_1), \quad (a_2'' \times d_1', d_1', +1/q_1),$$

$$(a_2' \times c_2', a_2', -1), \quad (b_2' \times c_2'', b_2', -1), \quad (a_1' \times c_2', c_2', +1/p_2), \quad (a_1'' \times d_2', d_2', +1/q_2),$$

followed by the following set of additional $2(n - 2)$ Luttinger surgeries

$$(b_1' \times c_3', c_3', -1/p_3), \quad (b_2' \times d_3', d_3', -1/q_3),$$

$$\cdots \cdots,$n

$$(b_1' \times c_n', c_n', -1/p_n), \quad (b_2' \times d_n', d_n', -1/q_n)$$
Lagrangian tori $a'_i \times c'_j$ and $a''_i \times d''_j$
Building Blocks

Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$

$Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)$

$e(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) = 4n - 4,$

$\sigma(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) = 0,$

$b_1(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) = 0$

$\pi_1(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n))$

$\pi_1(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n))$ is generated by loops $a_i, b_i, c_j, d_j$ $(i = 1, 2)$ and $j = 1, \ldots, n$ having the following relations:

\[
\begin{align*}
[a_1, a_2] &= n, & [b_1, a_1] &= 0, & [b_1, a_2] &= 0, \\
[b_1, b_2] &= b_2, & [c_1, b_1] &= 0, & [c_1, b_2] &= 0, \\
[c_1, c_2] &= c_2, & [d_1, c_1] &= 0, & [d_1, c_2] &= 0, \\
[d_1, d_2] &= d_1, & [e_1, d_1] &= 0, & [e_1, d_2] &= 0,
\end{align*}
\]
\( Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \)

\[
e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 4n - 4,
\]
\[
\sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0,
\]
\[
b_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0
\]

\[
\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))
\]
\[
\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \text{ is generated by loops } a_i, b_i, c_j, d_j, \ (i = 1, 2) \text{ and } j = 1, \ldots, n \text{ having the following relations :}
\]
Building Blocks

Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$

\[ Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \]

\[ e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 4n - 4, \]
\[ \sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0, \]
\[ b_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0 \]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \]
\[ \pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \text{ is generated by loops } a_i, b_i, c_j, d_j \text{, } (i = 1, 2) \text{ and } j = 1, \ldots, n \text{ having the following relations:} \]

\[
\begin{align*}
[a_1^{-1}, d_1] &= b_1, & [a_1^{-1}, b_2] &= d_1^0, & [a_1, c_1] &= 1, & [a_2, c_1] &= 1, & [b_1, c_1] &= 1, & [a_2, c_2] &= 1, & [a_1, d_1] &= 1, & [b_2, c_2] &= 1, \\
[a_2^{-1}, d_2] &= b_1, & [a_2^{-1}, b_2] &= d_2^0, & [a_1, c_2] &= 1, & [a_2, c_2] &= 1, & [b_2, c_2] &= 1, & [a_2, d_2] &= 1, & [b_1, c_2] &= 1, \\
[b_1^{-1}, d_1^{-1}] &= a_1, & [b_1^{-1}, b_2^{-1}] &= c_1^0, & [a_1, d_1] &= 1, & [a_1, b_1] &= 1, & [b_2, c_2] &= 1, & \Pi_{j=1}^{n} [c_j, d_j] &= 1, \\
[b_2^{-1}, d_2^{-1}] &= a_2, & [b_2^{-1}, b_2^{-1}] &= c_2^0, & \cdots & \cdots & \cdots & \cdots & \cdots, \\
[b_1^{-1}, d_3^{-1}] &= c_3^0, & \cdots & \cdots & \cdots & \cdots & \cdots, \\
[b_2^{-1}, d_3^{-1}] &= c_3^0, & \cdots & \cdots & \cdots & \cdots & \cdots, \\
[a_2^{-1}, d_3^{-1}] &= c_3^0, & \cdots & \cdots & \cdots & \cdots & \cdots, \\
[a_2^{-1}, d_n^{-1}] &= c_n^0, & \cdots & \cdots & \cdots & \cdots & \cdots, \\
[a_1^{-1}, d_n^{-1}] &= c_n^0, & \cdots & \cdots & \cdots & \cdots & \cdots.
\end{align*}
\]
Building Blocks

\[ Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \]

\[
e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 4n - 4,
\]
\[
\sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0,
\]
\[
b_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0
\]

\[
\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))
\]

\[
\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \text{ is generated by loops } a_i, b_i, c_j, d_j, \ (i = 1, 2) \text{ and } j = 1, \ldots, n \text{ having the following relations :}
\]
\[
[a_1^{-1}, d_1] = b_1,
[a_2^{-1}, d_2] = b_2,
[b_1^{-1}, d_1^{-1}] = a_1,
[b_2^{-1}, d_2^{-1}] = a_2,
[a_1^{-1}, d_3^{-1}] = c_3^{p_3},
[a_2^{-1}, c_3^{-1}] = d_3^{q_3},
[b_1, c_3] = 1,
[b_2, d_3] = 1
\]
\[
\cdots
\]
\[
[a_1^{-1}, d_n^{-1}] = c_n^{p_n},
[a_2^{-1}, c_n^{-1}] = d_n^{q_n},
[b_1, c_n] = 1,
[b_2, d_n] = 1
\]
Building Blocks

Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$

\[ Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n) \]

\[ e(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) = 4n - 4, \]
\[ \sigma(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) = 0, \]
\[ b_1(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) = 0 \]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) \]

$\pi_1(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n))$ is generated by loops $a_i, b_i, c_j, d_j, \ (i = 1, 2)$ and $j = 1, \ldots, n$ having the following relations:

\[
\begin{align*}
[a_1^{-1}, d_1] &= b_1, & [c_1^{-1}, b_2] &= d_1^{q_1}, & [a_1, c_1] &= 1, & [a_2, c_1] &= 1, & [b_1, c_1] &= 1, \\
[a_2^{-1}, d_2] &= b_2, & [c_2^{-1}, b_1] &= d_2^{q_2}, & [a_1, c_2] &= 1, & [a_2, c_2] &= 1, & [b_2, c_2] &= 1, \\
[b_1^{-1}, d_1^{-1}] &= a_1, & [d_1^{-1}, b_2^{-1}] &= c_1^{p_1}, & [a_1, d_2] &= 1, & [a_2, d_1] &= 1, & \prod_{j=1}^{n} [c_j, d_j] &= 1, \\
[b_2^{-1}, d_2^{-1}] &= a_2, & [d_2^{-1}, b_1^{-1}] &= c_2^{p_2}, & [a_1, b_1] &= 1, & [a_2, b_2] &= 1, & \prod_{j=1}^{n} [c_j, d_j] &= 1, \\
[a_1^{-1}, d_3^{-1}] &= c_3^{p_3}, & [a_2^{-1}, c_3^{-1}] &= d_3^{q_3}, & [b_1, c_3] &= 1, & [b_2, d_3] &= 1, \\
\cdots & & \cdots & & \cdots & & \cdots \\
[a_1^{-1}, d_n^{-1}] &= c_n^{p_n}, & [a_2^{-1}, c_n^{-1}] &= d_n^{q_n}, & [b_1, c_n] &= 1, & [b_2, d_n] &= 1.
\end{align*}
\]
Building Blocks

Luttinger surgeries on product manifolds \( \Sigma_n \times \Sigma_2 \)

\[
Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)
\]

\[
e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 4n - 4,
\]
\[
\sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0,
\]
\[
b_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0
\]

\[
\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))
\]

\(\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))\) is generated by loops \(a_i, b_i, c_j, d_j\), \(i = 1, 2\) and \(j = 1, \ldots, n\) having the following relations:

\[
[a_1^{-1}, d_1] = b_1,
\]
\[
[a_2^{-1}, d_2] = b_2,
\]
\[
[b_1^{-1}, d_1^{-1}] = a_1,
\]
\[
[b_2^{-1}, d_2^{-1}] = a_2,
\]
\[
[c_1^{-1}, b_2] = d_1^{q_1},
\]
\[
[c_2^{-1}, b_1] = d_2^{q_2},
\]
\[
[d_1^{-1}, b_2^{-1}] = c_1^{p_1},
\]
\[
[d_2^{-1}, b_1^{-1}] = c_2^{p_2},
\]
\[
[a_1, c_1] = 1,
\]
\[
[a_2, c_1] = 1,
\]
\[
[a_1, c_2] = 1,
\]
\[
[a_2, c_2] = 1,
\]
\[
[b_1, c_1] = 1,
\]
\[
[b_2, c_2] = 1,
\]
\[
\Pi_{j=1}^n [c_j, d_j] = 1,
\]
\[
[a_1^{-1}, d_3^{-1}] = c_3^{p_3},
\]
\[
[a_2^{-1}, d_3^{-1}] = d_3^{q_3},
\]
\[
[b_1, c_3] = 1,
\]
\[
[b_2, d_3] = 1,
\]
\[
[a_1^{-1}, d_4^{-1}] = c_4^{p_4},
\]
\[
[a_2^{-1}, c_4^{-1}] = d_4^{q_4},
\]
\[
[b_1, c_4] = 1,
\]
\[
[b_2, d_4] = 1.
\]
Building Blocks

Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$

\[ Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \]

\[ e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 4n - 4, \]
\[ \sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0, \]
\[ b_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0 \]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \]

$\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$ is generated by loops $a_i, b_i, c_j, d_j$, $(i = 1, 2)$ and $j = 1, \ldots, n$ having the following relations:

\[
\begin{align*}
[a_1^{-1}, d_1] &= b_1, & [c_1^{-1}, b_2] &= d_1^{q_1}, & [a_1, c_1] &= 1, & [a_2, c_1] &= 1, & [b_1, c_1] &= 1, \\
[a_2^{-1}, d_2] &= b_2, & [c_2^{-1}, b_1] &= d_2^{q_2}, & [a_1, c_2] &= 1, & [a_2, c_2] &= 1, & [b_2, c_2] &= 1, \\
[b_1^{-1}, d_1^{-1}] &= a_1, & [d_1^{-1}, b_2^{-1}] &= c_1^{p_1}, & [a_1, d_2] &= 1, & [a_2, d_1] &= 1, & \Pi_{j=1}^n[c_j, d_j] &= 1, \\
[b_2^{-1}, d_2^{-1}] &= a_2, & [d_2^{-1}, b_1^{-1}] &= c_2^{p_2}, & [a_1, b_1] &= 1, & [a_2, b_2] &= 1, & \Pi_{j=1}^n[c_j, d_j] &= 1,
\end{align*}
\]
Building Blocks Luttinger surgeries on product manifolds \( \Sigma_n \times \Sigma_2 \)

\[ Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \]

\[ e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 4n - 4, \]
\[ \sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0, \]
\[ b_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0 \]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \] is generated by loops \( a_i, b_i, c_j, d_j \), \( (i = 1, 2) \) and \( j = 1, \ldots, n \) having the following relations:

\[
\begin{align*}
[a_1^{-1}, d_1] &= b_1, & [c_1^{-1}, b_2] &= d_1^{q_1}, & [a_1, c_1] &= 1, & [a_2, c_1] &= 1, & [b_1, c_1] &= 1, \\
[a_2^{-1}, d_2] &= b_2, & [c_2^{-1}, b_1] &= d_2^{q_2}, & [a_1, c_2] &= 1, & [a_2, c_2] &= 1, & [b_2, c_2] &= 1, \\
[b_1^{-1}, d_1^{-1}] &= a_1, & [d_1^{-1}, b_2^{-1}] &= c_1^{p_1}, & [a_1, d_2] &= 1, & [a_2, d_1] &= 1, & \Pi_{j=1}^{n} [c_j, d_j] &= 1, \\
[b_2^{-1}, d_2^{-1}] &= a_2, & [d_2^{-1}, b_1^{-1}] &= c_2^{p_2}, & [a_1, b_1] &= 1, & [a_2, b_2] &= 1, & & \\
[a_1^{-1}, d_3^{-1}] &= c_3^{p_3}, & [a_2^{-1}, c_3^{-1}] &= d_3^{q_3}, & [b_1, c_3] &= 1, & & & \\
[a_1^{-1}, d_n^{-1}] &= c_n^{p_n}, & [a_2^{-1}, c_n^{-1}] &= d_n^{q_n}, & [b_1, c_n] &= 1, & & & \\
& & & [b_2, d_3] &= 1, & & & \\
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\end{align*}
\]
$Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$

e$(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 4n - 4,$

$\sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0,$

$b_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0$

$\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$

$\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$ is generated by loops $a_i, b_i, c_j, d_j$ (i = 1, 2) and $j = 1, \ldots, n$ having the following relations:

$[a_1^{-1}, d_1] = b_1,$

$[a_2^{-1}, d_2] = b_2,$

$[b_1^{-1}, d_1^{-1}] = a_1,$

$[b_2^{-1}, d_2^{-1}] = a_2,$

$[c_1^{-1}, b_2] = d_1^{q_1},$ 

$[c_2^{-1}, b_1] = d_2^{q_2},$ 

$[d_1^{-1}, b_2^{-1}] = c_1^{p_1},$ 

$[d_2^{-1}, b_1^{-1}] = c_2^{p_2},$

$[a_1, c_1] = 1,$

$[a_2, c_1] = 1,$

$[a_1, c_2] = 1,$

$[a_2, c_2] = 1,$

$[b_1, c_1] = 1,$

$[b_2, c_2] = 1,$

$\Pi_{j=1}^n[c_j, d_j] = 1,$

$[b_1, c_2] = 1,$

$[b_2, c_1] = 1,$

$[b_2, d_3] = 1,$

$[b_2, d_1] = 1,$

$[b_1, d_2] = 1,$

$[b_1, d_1] = 1,$

$[b_2, d_2] = 1,$

$[b_2, d_1] = 1.$
\[ Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n) \]

\[
e(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) = 4n - 4,
\]
\[
\sigma(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) = 0,
\]
\[
b_1(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) = 0
\]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) \]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \ldots, 1/p_n, 1/q_n)) \text{ is generated by loops } a_i, b_i, c_j, d_j, \ (i = 1, 2) \text{ and } j = 1, \ldots, n \text{ having the following relations :} \]

\[
[a_1^{-1}, d_1] = b_1, \quad [c_1^{-1}, b_2] = d_1^{q_1}, \quad [a_1, c_1] = 1, \quad [a_2, c_1] = 1, \quad [b_1, c_1] = 1, \quad [b_2, c_2] = 1, \\
[a_2^{-1}, d_2] = b_2, \quad [c_2^{-1}, b_1] = d_2^{q_2}, \quad [a_1, c_2] = 1, \quad [a_2, c_2] = 1, \quad [b_2, c_2] = 1, \\
[b_1^{-1}, d_1^{-1}] = a_1, \quad [d_1^{-1}, b_2^{-1}] = c_1^{p_1}, \quad [a_1, d_2] = 1, \quad [a_2, d_1] = 1, \quad \Pi_{j=1}^{n} [c_j, d_j] = 1, \\
[b_1^{-1}, d_1^{-1}] = a_2, \quad [d_2^{-1}, b_1^{-1}] = c_2^{p_2}, \quad [a_1, b_1] = 1, \quad [a_2, b_2] = 1, \\
[\cdots] \quad [d_1^{-1}, c_1^{-1}] = d_1^{q_3}, \quad [a_1, c_3] = 1, \quad [b_2, d_3] = 1, \\
[\cdots] \quad [d_2^{-1}, c_2^{-1}] = d_2^{q_3}, \quad [a_2, c_3] = 1, \quad [b_2, d_3] = 1, \\
[\cdots] \quad [a_1^{-1}, d_n^{-1}] = c_n^{p_n}, \quad [a_2^{-1}, c_n^{-1}] = d_n^{q_n}, \quad [b_1, c_n] = 1, \quad [b_2, d_n] = 1.
\]
\[ Y_n \left( \frac{1}{p_1}, \frac{1}{q_1}, \cdots, \frac{1}{p_n}, \frac{1}{q_n} \right) \]

\[ e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 4n - 4, \]
\[ \sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0, \]
\[ b_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0 \]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \text{ is generated by loops } a_i, b_i, c_j, d_j, \text{ (} i = 1, 2 \text{) and } j = 1, \ldots, n \text{) having the following relations:} \]

\[
\begin{align*}
[a_1^{-1}, d_1] &= b_1, \\
[a_2^{-1}, d_2] &= b_2, \\
[b_1^{-1}, d_1^{-1}] &= a_1, \\
[b_2^{-1}, d_2^{-1}] &= a_2, \\
[a_1^{-1}, d_3^{-1}] &= c_3^{p_3}, \\
\ldots \\
[a_1^{-1}, d_n^{-1}] &= c_n^{p_n}, \\
\end{align*}
\]

\[
\begin{align*}
[c_1^{-1}, b_2] &= d_1^{q_1}, \\
[c_2^{-1}, b_1] &= d_2^{q_2}, \\
[d_1^{-1}, b_2^{-1}] &= c_1^{p_1}, \\
[d_2^{-1}, b_1^{-1}] &= c_2^{p_2}, \\
[a_1, c_1] &= 1, \\
[a_2, c_1] &= 1, \\
[a_1, c_2] &= 1, \\
[a_2, c_2] &= 1, \\
[a_1, d_2] &= 1, \\
[a_2, d_1] &= 1, \\
[a_1, b_1] &= 1, \\
[a_2, b_2] &= 1, \\
[b_1, c_1] &= 1, \\
[b_2, c_2] &= 1, \\
[\Pi_{j=1}^{n} c_j, d_j] &= 1, \\
\end{align*}
\]

\[
\begin{align*}
[a_2^{-1}, c_3^{-1}] &= d_3^{q_3}, \\
[b_1, c_3] &= 1, \\
[b_2, d_3] &= 1, \\
\ldots \\
[a_2^{-1}, c_n^{-1}] &= d_n^{q_n}, \\
[b_1, c_n] &= 1, \\
[b_2, d_n] &= 1. \\
\end{align*}
\]
$Y_n(1/p_1, 1/q_1, \cdots , 1/p_n, 1/q_n)$

$e(Y_n(1/p_1, 1/q_1, \cdots , 1/p_n, 1/q_n)) = 4n - 4$,  
$\sigma(Y_n(1/p_1, 1/q_1, \cdots , 1/p_n, 1/q_n)) = 0$,  
$b_1(Y_n(1/p_1, 1/q_1, \cdots , 1/p_n, 1/q_n)) = 0$

$\pi_1(Y_n(1/p_1, 1/q_1, \cdots , 1/p_n, 1/q_n))$ is generated by loops $a_i, b_i, c_j, d_j \ (i = 1, 2)$ and $j = 1, \ldots, n$ having the following relations:

$\begin{align*}
[a_1^{-1}, d_1] &= b_1, \\
[a_2^{-1}, d_2] &= b_2, \\
[b_1^{-1}, d_1^{-1}] &= a_1, \\
[b_2^{-1}, d_2^{-1}] &= a_2, \\
[a_1^{-1}, d_3^{-1}] &= c_3^{p_3}, \\
\ldots \\
[a_1^{-1}, d_n^{-1}] &= c_n^{p_n}, \\
[a_2^{-1}, c_3^{-1}] &= d_3^{q_3}, \\
\ldots \\
[a_2^{-1}, c_n^{-1}] &= d_n^{q_n},
\end{align*}$

$[a_1, c_1] = 1$,  
$[a_2, c_1] = 1$,  
$[b_1, c_1] = 1$,  
$[a_1, c_2] = 1$,  
$[a_2, c_2] = 1$,  
$[b_2, c_2] = 1$,  
$[a_1, d_1] = 1$,  
$[a_2, d_1] = 1$,  
$\Pi_{j=1}^{n} [c_j, d_j] = 1$,
Building Blocks
Luttinger surgeries on product manifolds \( \Sigma_n \times \Sigma_2 \)

\[ Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \]

\[
e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 4n - 4,\]
\[
\sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0,\]
\[
b_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0
\]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \]

\( \pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \) is generated by loops \( a_i, b_i, c_j, d_j, (i = 1, 2) \) and \( j = 1, \ldots, n \) having the following relations:

\[
[a_1^{-1}, d_1] = b_1, \quad [a_2^{-1}, d_2] = b_2, \quad [b_1^{-1}, d_1^{-1}] = a_1, \quad [b_2^{-1}, d_2^{-1}] = a_2,
\]
\[
[c_1^{-1}, b_2] = d_1^{q_1}, \quad [c_2^{-1}, b_1] = d_2^{q_2}, \quad [d_1^{-1}, b_2^{-1}] = c_1^{p_1}, \quad [d_2^{-1}, b_1^{-1}] = c_2^{p_2},
\]
\[
[a_1, c_1] = 1, \quad [a_2, c_1] = 1, \quad [a_1, c_2] = 1, \quad [a_2, c_2] = 1, \quad [a_1, d_1] = 1, \quad [a_2, d_1] = 1, \quad [a_1, b_1] = 1, \quad [a_2, b_1] = 1,
\]
\[
[b_1, c_1] = 1, \quad [b_2, c_2] = 1, \quad \Pi_{j=1}^{n} [c_j, d_j] = 1, \quad [b_1, c_3] = 1, \quad [b_2, d_3] = 1, \quad [b_1, c_n] = 1, \quad [b_2, d_n] = 1.
\]
Building Blocks

Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$

\[ Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \]

\[ e(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 4n - 4, \]
\[ \sigma(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0, \]
\[ b_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) = 0 \]

\[ \pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)) \]

$\pi_1(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n))$ is generated by loops $a_i, b_i, c_j, d_j$, $(i = 1, 2)$ and $(j = 1, \ldots, n)$ having the following relations:

\[ [a_1^{-1}, d_1] = b_1, \quad [c_1^{-1}, b_2] = d_1^{q_1}, \quad [a_1, c_1] = 1, \quad [a_2, c_1] = 1, \quad [b_1, c_1] = 1, \]
\[ [a_2^{-1}, d_2] = b_2, \quad [c_2^{-1}, b_1] = d_2^{q_2}, \quad [a_1, c_2] = 1, \quad [a_2, c_2] = 1, \quad [b_2, c_2] = 1, \]
\[ [b_1^{-1}, d_1^{-1}] = a_1, \quad [d_1^{-1}, b_2^{-1}] = c_1^{p_1}, \quad [a_1, d_2] = 1, \quad [a_2, d_1] = 1, \quad \Pi_{j=1}^{n} [c_j, d_j] = 1, \]
\[ [d_2^{-1}, b_1^{-1}] = c_2^{p_2}, \quad [a_1, b_1] = 1, \quad [a_2, b_2] = 1, \]
\[ [a_1^{-1}, d_3^{-1}] = c_3^{p_3}, \quad [a_2^{-1}, c_3^{-1}] = d_3^{q_3}, \quad [b_1, c_3] = 1, \quad [b_2, d_3] = 1, \]
\[ \cdots \]
\[ [a_1^{-1}, d_n^{-1}] = c_n^{p_n}, \quad [a_2^{-1}, c_n^{-1}] = d_n^{q_n}, \quad [b_1, c_n] = 1, \quad [b_2, d_n] = 1. \]
$Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$

Since the surfaces $\{pt\} \times \Sigma_2$ and $\Sigma_n \times \{pt\}$ in $\Sigma_n \times \Sigma_2$ are not affected by the above Luttinger surgeries, they descend to surfaces in $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$. Let us denote these symplectic submanifolds by $\Sigma_2$ and $\Sigma_n$.

Notice that we have $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$ and $[\Sigma_2] \cdot [\Sigma_n] = 1$
Since the surfaces \{pt\} \times \Sigma_2 and \Sigma_n \times \{pt\} in \Sigma_n \times \Sigma_2 are not affected by the above Luttinger surgeries, they descend to surfaces in \(Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)\). Let us denote these symplectic submanifolds by \(\Sigma_2\) and \(\Sigma_n\). Notice that we have \([\Sigma_2]^2 = [\Sigma_n]^2 = 0\) and \([\Sigma_2] \cdot [\Sigma_n] = 1\).
Since the surfaces \( \{pt\} \times \Sigma_2 \) and \( \Sigma_n \times \{pt\} \) in \( \Sigma_n \times \Sigma_2 \) are not affected by the above Luttinger surgeries, they descend to surfaces in \( Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n) \).

Let us denote these symplectic submanifolds by \( \Sigma_2 \) and \( \Sigma_n \).

Notice that we have \( [\Sigma_2]^2 = [\Sigma_n]^2 = 0 \) and \( [\Sigma_2] \cdot [\Sigma_n] = 1 \).
Building Blocks

Luttinger surgeries on product manifolds $\Sigma_n \times \Sigma_2$

$Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$

Since the surfaces $\{pt\} \times \Sigma_2$ and $\Sigma_n \times \{pt\}$ in $\Sigma_n \times \Sigma_2$ are not affected by the above Luttinger surgeries, they descend to surfaces in $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$. Let us denote these symplectic submanifolds by $\Sigma_2$ and $\Sigma_n$.

Notice that we have $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$ and $[\Sigma_2] \cdot [\Sigma_n] = 1$.
Let us denote these symplectic submanifolds by $\Sigma_2$ and $\Sigma_n$. Notice that we have $[\Sigma_2]^2 = [\Sigma_n]^2 = 0$ and $[\Sigma_2] \cdot [\Sigma_n] = 1$. 

Since the surfaces $\{pt\} \times \Sigma_2$ and $\Sigma_n \times \{pt\}$ in $\Sigma_n \times \Sigma_2$ are not affected by the above Luttinger surgeries, they descend to surfaces in $Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$. 

$Y_n(1/p_1, 1/q_1, \cdots, 1/p_n, 1/q_n)$
Let us fix a quadruple of integers \( n \geq 2, m \geq 1, p \geq 1 \) and \( q \geq 1 \).
Let \( Y_n(1/p, m/q) \) denote smooth 4-manifold obtained by performing the following \( 2n \) torus surgeries on \( \Sigma_n \times \mathbb{T}^2 \):

\[
\begin{align*}
(a_1, c_1, -1), & (a_2, c_2, -1), & \cdots, & (a_n, c_n, -1), \\
(a_1, c_1', c_1', +1/p), & (a_2, c_2', c_2', +1/p), & \cdots, & (a_n, c_n', c_n', +1/p), \\
(b_1, d_1, -1), & (b_2, d_2, -1), & \cdots, & (b_n, d_n, -1), \\
(b_1, d_1', d_1', +m/q), & (b_2, d_2', d_2', +m/q), & \cdots, & (b_n, d_n', d_n', +m/q).
\end{align*}
\]

where \( a_i, b_i (i = 1, 2, \cdots, n) \) and \( c, d \) denote the standard generators \( \pi_1(\Sigma_{2n}) \) and \( \pi_1(\mathbb{T}^2) \), respectively.

Let \( \Sigma_n' \subset Y_n(1/p, m/q) \) denotes a genus \( n \) surface that descend from \( \Sigma_n \times pt \) in \( \Sigma_n \times \mathbb{T}^2 \).
Let us fix a quadruple of integers $n \geq 2$, $m \geq 1$, $p \geq 1$ and $q \geq 1$. Let $Y_n(1/p, m/q)$ denote smooth 4-manifold obtained by performing the following $2n$ torus surgeries on $\Sigma_n \times \mathbb{T}^2$:

$$
\begin{align*}
(a'_{i1} \times c'_{\alpha}, a'_{i1}, -1), \\
(a'_{i2} \times c'_{\alpha}, a'_{i2}, -1), \\
\vdots \\
(a'_{in-1} \times c'_{\alpha}, a'_{in-1}, -1), \\
(a'_{in} \times c'_{\alpha}, c'_{\alpha}, \frac{1}{p} + 1), \\
(b'_{i1} \times c''_{\beta}, b'_{i1}, -1), \\
(b'_{i2} \times c''_{\beta}, b'_{i2}, -1), \\
\vdots \\
(b'_{in-1} \times c''_{\beta}, b'_{in-1}, -1), \\
(a''_{in} \times d'_{\gamma}, d'_{\gamma}, \frac{m}{q}).
\end{align*}
$$

where $a_i, b_i$ ($i = 1, 2, \cdots, n$) and $c, d$ denote the standard generators $\pi_1(\Sigma_{2n})$ and $\pi_1(\mathbb{T}^2)$, respectively.

Let $\Sigma'_n \subset Y_n(1/p, l/q)$ denotes a genus $n$ surface that desend from $\Sigma_n \times pt$ in $\Sigma_n \times \mathbb{T}^2$. 
Let us fix a quadruple of integers $n \geq 2$, $m \geq 1$, $p \geq 1$ and $q \geq 1$. Let $Y_n(1/p, m/q)$ denote smooth 4-manifold obtained by performing the following $2n$ torus surgeries on $\Sigma_n \times T^2$:

\[
(a'_1 \times c', a'_1, -1), \\
(a'_2 \times c', a'_2, -1), \\
\vdots, \\
(a'_{n-1} \times c', a'_{n-1}, -1), \\
(a'_n \times c', c', +1/p), \\
\]

\[
(b'_1 \times c'', b'_1, -1), \\
(b'_2 \times c'', b'_2, -1), \\
\vdots, \\
(b''_{n-1} \times c'', b''_{n-1}, -1), \\
(a''_n \times d', d', +m/q) .
\]

where $a_i, b_i$ ($i = 1, 2, \cdots, n$) and $c, d$ denote the standard generators $\pi_1(\Sigma_{2n})$ and $\pi_1(T^2)$, respectively.
Let us fix a quadruple of integers $n \geq 2$, $m \geq 1$, $p \geq 1$ and $q \geq 1$. Let $Y_n(1/p, m/q)$ denote smooth 4-manifold obtained by performing the following $2n$ torus surgeries on $\Sigma_n \times T^2$:

\[
\begin{align*}
(a'_1 \times c', a'_1, -1), \\
(a'_2 \times c', a'_2, -1), \\
\vdots, \\
(a'_{n-1} \times c', a'_{n-1}, -1), \\
(a'_n \times c', c', +1/p), \\
(b'_1 \times c'', b'_1, -1), \\
(b'_2 \times c'', b'_2, -1), \\
\vdots, \\
(b'_{n-1} \times c'', b'_{n-1}, -1), \\
(a''_n \times d', d', +m/q).
\end{align*}
\]

where $a_i, b_i$ $(i = 1, 2, \ldots, n)$ and $c, d$ denote the standard generators $\pi_1(\Sigma_{2n})$ and $\pi_1(T^2)$, respectively.
Let us fix a quadruple of integers \( n \geq 2, m \geq 1, p \geq 1 \) and \( q \geq 1 \).
Let \( Y_n(1/p, m/q) \) denote smooth 4-manifold obtained by performing the following \( 2n \) torus surgeries on \( \Sigma_n \times \mathbb{T}^2 \):

\[
(a_1' \times c', a_1', -1), \quad (b_1' \times c'', b_1', -1),
(a_2' \times c', a_2', -1), \quad (b_2' \times c'', b_2', -1),
\ldots, \quad \ldots,
(a_n' \times c', a_{n-1}' \times c', -1), \quad (b_{n-1}' \times c'', b_{n-1}', -1),
(a_n' \times c', +1/p), \quad (a_n'' \times d', +m/q).
\]

where \( a_i, b_i \) \( (i = 1, 2, \ldots, n) \) and \( c, d \) denote the standard generators \( \pi_1(\Sigma_{2n}) \) and \( \pi_1(\mathbb{T}^2) \), respectively.
Let us fix a quadruple of integers \( n \geq 2, \ m \geq 1, \ p \geq 1 \) and \( q \geq 1 \).
Let \( Y_n(1/p, \ m/q) \) denote smooth 4-manifold obtained by performing the following \( 2n \) torus surgeries on \( \Sigma_n \times \mathbb{T}^2 \):

\[
(a'_1 \times c', a'_1, -1), \quad (b'_1 \times c'', b'_1, -1), \\
(a'_2 \times c', a'_2, -1), \quad (b'_2 \times c'', b'_2, -1), \\
\vdots, \quad \vdots, \\
(a'_{n-1} \times c', a'_{n-1}, -1), \quad (b'_{n-1} \times c'', b'_{n-1}, -1), \\
(a'_n \times c', c', +1/p), \quad (a''_n \times d', d', +m/q).
\]

where \( a_i, b_i (i = 1, 2, \ldots, n) \) and \( c, d \) denote the standard generators \( \pi_1(\Sigma_{2n}) \) and \( \pi_1(\mathbb{T}^2) \), respectively.
Let us fix a quadruple of integers $n \geq 2$, $m \geq 1$, $p \geq 1$ and $q \geq 1$. Let $Y_n(1/p, m/q)$ denote smooth 4-manifold obtained by performing the following $2n$ torus surgeries on $\Sigma_n \times \mathbb{T}^2$:

$$(a'_1 \times c', a'_1, -1), \quad (b'_1 \times c'', b'_1, -1),$$
$$(a'_2 \times c', a'_2, -1), \quad (b'_2 \times c'', b'_2, -1),$$
$$\cdots,$n
$$(a'_{n-1} \times c', a'_{n-1}, -1), \quad (b'_{n-1} \times c'', b'_{n-1}, -1),$$
$$(a'_n \times c', c', +1/p), \quad (a''_n \times d', d', +m/q).$$

where $a_i, b_i \ (i = 1, 2, \cdots, n)$ and $c, d$ denote the standard generators $\pi_1(\Sigma_{2n})$ and $\pi_1(\mathbb{T}^2)$, respectively.
Let us fix a quadruple of integers $n \geq 2, m \geq 1, p \geq 1$ and $q \geq 1$. Let $Y_n(1/p, m/q)$ denote smooth 4-manifold obtained by performing the following $2n$ torus surgeries on $\Sigma_n \times \mathbb{T}^2$:

$$(a'_1 \times c', a'_1, -1),$$
$$(a'_2 \times c', a'_2, -1),$$
$$\ldots ,$$
$$(a'_{n-1} \times c', a'_{n-1}, -1),$$
$$(a'_n \times c', c', +1/p),$$

$$(b'_1 \times c'', b'_1, -1),$$
$$(b'_2 \times c'', b'_2, -1),$$
$$\ldots ,$$
$$(b'_{n-1} \times c'', b'_{n-1}, -1),$$
$$(a''_n \times d', d', +m/q).$$

where $a_i, b_i \ (i = 1, 2, \ldots, n)$ and $c, d$ denote the standard generators $\pi_1(\Sigma_{2n})$ and $\pi_1(\mathbb{T}^2)$, respectively.
Let us fix a quadruple of integers \( n \geq 2, m \geq 1, p \geq 1 \) and \( q \geq 1 \).

Let \( Y_n(1/p, m/q) \) denote smooth 4-manifold obtained by performing the following \( 2n \) torus surgeries on \( \Sigma_n \times \mathbb{T}^2 \):

\[
(a'_1 \times c', a'_1, -1), \\
(a'_2 \times c', a'_2, -1), \\
\ldots, \\
(a'_n-1 \times c', a'_n-1, -1), \\
(a'_n \times c', c', +1/p), \\
(b'_1 \times c'', b'_1, -1), \\
(b'_2 \times c'', b'_2, -1), \\
\ldots, \\
(b'_{n-1} \times c'', b'_{n-1}, -1), \\
(a''_n \times d', d', +m/q).
\]

where \( a_i, b_i \ (i = 1, 2, \ldots, n) \) and \( c, d \) denote the standard generators \( \pi_1(\Sigma_{2n}) \) and \( \pi_1(\mathbb{T}^2) \), respectively.

Let \( \Sigma'_n \subset Y_n(1/p, l/q) \) denotes a genus \( n \) surface that desend from \( \Sigma_n \times pt \) in \( \Sigma_n \times \mathbb{T}^2 \).
The fundamental group of $Y_n(1/p, m/q)$ is generated by loops $a_i, b_i$ ($i = 1, 2, \cdots, m$) and $c, d$ having the following relations in $\pi_1(Y_n(1/p, m/q))$:

\[
\begin{align*}
[a_1, d] &= b_1, \\
[a_2, d] &= b_2, \\
&\quad \vdots \\
[a_{n-1}, d] &= b_{n-1}, \\
[a_n, d] &= b_n, \\
[a_{1}, c] &= 1, \\
[a_{2}, c] &= 1, \\
&\quad \vdots \\
[a_{n-1}, c] &= 1, \\
[a_{n}, c] &= 1.
\end{align*}
\]
\[ \pi_1(Y_n(1/p, m/q)) \]

The fundamental group of \( Y_n(1/p, m/q) \) is generated by loops \( a_i, b_i \) (\( i = 1, 2, \cdots, m \)) and \( c, d \) having the following relations in \( \pi_1(Y_n(1/p, m/q)) \):

\[
\begin{align*}
[a_1, c] &= 1, \\
[b_1^{-1}, d^{-1}] &= a_1, \\
[b_2^{-1}, d^{-1}] &= a_2, \\
&\vdots \\
[b_{n-1}^{-1}, d^{-1}] &= a_{n-1}, \\
[a_{n-1}^{-1}, d] &= b_{n-1}, \\
[a_n^{-1}, d] &= b_n, \\
&\vdots \\
[a_{n-1}, c] &= 1, \\
[b_{n-1}, c] &= 1, \\
[a_n, c] &= 1, \\
[a_n, d] &= 1, \\
[d^{-1}, b_n^{-1}] &= c, \\
[c^{-1}, b_n]^{-m} &= d^q, \\
[a_1, b_1] \cdot [a_2, b_2] \cdots [a_n, b_n] &= 1, \\
[c, d] &= 1.
\end{align*}
\]
The fundamental group of $Y_n(1/p, m/q)$ is generated by loops $a_i, b_i$ ($i = 1, 2, \cdots, m$) and $c, d$ having the following relations in $\pi_1(Y_n(1/p, m/q))$:

\[
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
[b_2^{-1}, d^{-1}] &= a_2, \\
\cdots \\
[b_{n-1}^{-1}, d^{-1}] &= a_{n-1} \\
[a_1^{-1}, d] &= b_1, \\
[a_2^{-1}, d] &= b_2, \\
\cdots \\
[a_{n-1}^{-1}, d] &= b_{n-1}, \\
[a_1, c] &= 1, \\
[a_2, c] &= 1, \\
\cdots \quad [b_1, c] &= 1, \\
[b_2, c] &= 1, \\
\cdots \quad [a_{n-1}, c] &= 1, \\
[a_n, c] &= 1, \\
[a_n, d] &= 1, \\
[b_{n-1}, c] &= 1, \\
\end{align*}
\]

\[
\begin{align*}
[d^{-1}, b_n^{-1}] &= c^p, \\
[c^{-1}, b_n]^{-m} &= d^q, \\
[a_1, b_1] \cdot [a_2, b_2] \cdots [a_n, b_n] &= 1, \\
[c, d] &= 1.
\end{align*}
\]
\[ \pi_1(Y_n(1/p, m/q)) \]

The fundamental group of \( Y_n(1/p, m/q) \) is generated by loops \( a_i, b_i \) \((i = 1, 2, \ldots, m)\) and \( c, d \) having the following relations in \( \pi_1(Y_n(1/p, m/q)) \):

\[
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, & [a_1^{-1}, d] &= b_1, & [a_1, c] &= 1, \\
[b_2^{-1}, d^{-1}] &= a_2, & [a_2^{-1}, d] &= b_2, & [a_2, c] &= 1, \\
&\vdots & & \vdots \\
[b_{n-1}^{-1}, d^{-1}] &= a_{n-1}, & [a_{n-1}^{-1}, d] &= b_{n-1}, & [a_{n-1}, c] &= 1, \\
[b_n^{-1}, d^{-1}] &= a_n \quad \text{(not shown)} & [a_n^{-1}, d] &= b_n, & [a_n, c] &= 1, \\
& & [a_n, d] &= 1, & [b_n, c] &= 1, \\
& & & [b_{n-1}, c] &= 1 \\
[d^{-1}, b_n^{-1}] &= c^p, & [c^{-1}, b_n]^{-m} &= d^q, \\
[a_1, b_1] \cdot [a_2, b_2] \cdots [a_n, b_n] &= 1, & [c, d] &= 1.
\]
The fundamental group of $Y_n(1/p, m/q)$ is generated by loops $a_i, b_i$ ($i = 1, 2, \cdots, m$) and $c, d$ having the following relations in $\pi_1(Y_n(1/p, m/q))$:

$$
egin{align*}
\left[ b_1^{-1}, d^{-1} \right] &= a_1, \\
\left[ b_2^{-1}, d^{-1} \right] &= a_2, \\
&\cdots \\
\left[ b_{n-1}^{-1}, d^{-1} \right] &= a_{n-1},
\end{align*}
$$

$$
\begin{align*}
\left[ a_1^{-1}, d \right] &= b_1, \\
\left[ a_2^{-1}, d \right] &= b_2, \\
&\cdots \\
\left[ a_{n-1}^{-1}, d \right] &= b_{n-1},
\end{align*}
$$

$$
\begin{align*}
\left[ a_1, c \right] &= 1, \\
\left[ a_2, c \right] &= 1, \\
&\cdots \\
\left[ a_n, c \right] &= 1, \\
\left[ a_1, d \right] &= 1,
\end{align*}
$$

$$
\begin{align*}
\left[ b_1, c \right] &= 1, \\
\left[ b_2, c \right] &= 1, \\
&\cdots \\
\left[ b_{n-1}, c \right] &= 1,
\end{align*}
$$

$$
\begin{align*}
\left[ b_1, d^{-1} \right] &= c^p, \\
\left[ b_2, d^{-1} \right] &= b_n^{-m} = d^q, \\
\left[ a_1, b_1 \right] \cdot \left[ a_2, b_2 \right] \cdots \left[ a_n, b_n \right] &= 1, \\
\left[ c, d \right] &= 1.
\end{align*}
$$
The fundamental group of $Y_n(1/p, m/q)$ is generated by loops $a_i, b_i$ ($i = 1, 2, \ldots, m$) and $c, d$ having the following relations in $\pi_1(Y_n(1/p, m/q))$:

$$
[b_1^{-1}, d^{-1}] = a_1, \quad \quad [a_1^{-1}, d] = b_1, \quad \quad [a_1, c] = 1, \\
[b_2^{-1}, d^{-1}] = a_2, \quad \quad [a_2^{-1}, d] = b_2, \quad \quad [a_2, c] = 1, \\
\vdots \quad \quad \vdots \quad \quad \vdots \\
[b_{n-1}^{-1}, d^{-1}] = a_{n-1}, \quad \quad [a_{n-1}^{-1}, d] = b_{n-1}, \quad \quad [a_{n-1}, c] = 1, \\
[a_n, d] = 1
$$

$$
[d^{-1}, b_n^{-1}] = c^q, \quad [c^{-1}, b_n]^{-m} = d^p, \\
[a_1, b_1] \cdot [a_2, b_2] \cdots [a_n, b_n] = 1, \quad [c, d] = 1.
$$
The fundamental group of $Y_n(1/p, m/q)$ is generated by loops $a_i, b_i$ ($i = 1, 2, \cdots, m$) and $c, d$ having the following relations in $\pi_1(Y_n(1/p, m/q))$:

\[
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
[b_2^{-1}, d^{-1}] &= a_2, \\
&\quad \cdots \\
[b_{n-1}^{-1}, d^{-1}] &= a_{n-1}, \\
[a_1^{-1}, d] &= b_1, \\
[a_2^{-1}, d] &= b_2, \\
&\quad \cdots \\
[a_{n-1}^{-1}, d] &= b_{n-1}, \\
[a_1, c] &= 1, \\
[a_2, c] &= 1, \\
&\quad \cdots \\
[a_{n-1}, c] &= 1, \\
[a_n, c] &= 1, \\
[a_n, d] &= 1, \\
[b_1, c] &= 1, \\
[b_2, c] &= 1, \\
&\quad \cdots \\
[b_{n-1}, c] &= 1
\end{align*}
\]
The fundamental group of $Y_n(1/p, m/q)$ is generated by loops $a_i, b_i \ (i = 1, 2, \cdots, m)$ and $c, d$ having the following relations in $\pi_1(Y_n(1/p, m/q))$:

\[
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
[b_2^{-1}, d^{-1}] &= a_2, \\
&\cdots \\
[b_{n-1}^{-1}, d^{-1}] &= a_{n-1}
\end{align*}
\]

\[
\begin{align*}
[a_1^{-1}, d] &= b_1, \\
[a_2^{-1}, d] &= b_2, \\
&\cdots, \\
[a_{n-1}^{-1}, d] &= b_{n-1},
\end{align*}
\]

\[
\begin{align*}
[a_1, c] &= 1, \\
[a_2, c] &= 1, \\
&\cdots, \\
[a_{n-1}, c] &= 1, \\
[a_n, c] &= 1, \\
[a_n, d] &= 1,
\end{align*}
\]

\[
\begin{align*}
[b_1, c] &= 1, \\
[b_2, c] &= 1, \\
&\cdots, \\
[b_{n-1}, c] &= 1
\end{align*}
\]

\[
\begin{align*}
[d^{-1}, b_n^{-1}] &= c^p, \\
[c^{-1}, b_n]^{-m} &= d^q,
\end{align*}
\]

\[
\begin{align*}
[a_1, b_1] \cdot [a_2, b_2] \cdots [a_n, b_n] &= 1, \\
[c, d] &= 1.
\end{align*}
\]
The fundamental group of $Y_n(1/p, m/q)$ is generated by loops $a_i, b_i$ ($i = 1, 2, \cdots, m$) and $c, d$ having the following relations in $\pi_1(Y_n(1/p, m/q))$:

$$[b_1^{-1}, d^{-1}] = a_1,$$
$$[b_2^{-1}, d^{-1}] = a_2,$$
$$\cdots$$
$$[b_{n-1}^{-1}, d^{-1}] = a_{n-1},$$

$$[a_1^{-1}, d] = b_1,$$
$$[a_2^{-1}, d] = b_2,$$
$$\cdots$$
$$[a_{n-1}^{-1}, d] = b_{n-1},$$

$$[a_1, c] = 1,$$
$$[a_2, c] = 1,$$
$$\cdots$$
$$[a_{n-1}, c] = 1,$$
$$[a_n, c] = 1,$$
$$[a_n, d] = 1,$$

$$[d^{-1}, b_n^{-1}] = c^p,$$
$$[c^{-1}, b_n]^{-m} = d^q,$$

$$[a_1, b_1] \cdot [a_2, b_2] \cdots [a_n, b_n] = 1, [c, d] = 1.$$
The fundamental group of \( Y_n(1/p, m/q) \) is generated by loops \( a_i, b_i \) (\( i = 1, 2, \cdots, m \)) and \( c, d \) having the following relations in \( \pi_1(Y_n(1/p, m/q)) \):

\[
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
[b_2^{-1}, d^{-1}] &= a_2, \\
&\cdots \\
[b_{n-1}^{-1}, d^{-1}] &= a_{n-1} \\
[a_1^{-1}, d] &= b_1, \\
[a_2^{-1}, d] &= b_2, \\
&\cdots \\
[a_{n-1}^{-1}, d] &= b_{n-1}, \\
[a_1, c] &= 1, \\
[a_2, c] &= 1, \\
&\cdots \\
[a_n, c] &= 1, \\
[a_1, c] &= 1, \\
[a_2, c] &= 1, \\
&\cdots \\
[a_n, c] &= 1, \\
[b_1, c] &= 1, \\
[b_2, c] &= 1, \\
&\cdots \\
[b_{n-1}, c] &= 1 \\
[d^{-1}, b_n^{-1}] &= c^p, \\
[c^{-1}, b_n]^{-m} &= d^q, \\
[a_1, b_1] \cdot [a_2, b_2] \cdots [a_n, b_n] &= 1, \\
[c, d] &= 1.
\end{align*}
\]
The fundamental group of \( Y_n(1/p, m/q) \) is generated by loops \( a_i, b_i \ (i = 1, 2, \cdots, m) \) and \( c, d \) having the following relations in \( \pi_1(Y_n(1/p, m/q)) \):

\[
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
[b_2^{-1}, d^{-1}] &= a_2, \\
\cdots \\
[b_{n-1}^{-1}, d^{-1}] &= a_{n-1}, \\
[a_1^{-1}, d] &= b_1, \\
[a_2^{-1}, d] &= b_2, \\
\cdots \\
[a_{n-1}^{-1}, d] &= b_{n-1}, \\
[a_1, c] &= 1, \\
[a_2, c] &= 1, \\
\cdots, \\
[a_{n-1}, c] &= 1, \\
[a_n, c] &= 1, \\
[a_n, d] &= 1, \\
[d^{-1}, b_1^{-1}] &= c^p, \\
[c^{-1}, b_n]^{-m} &= d^q, \\
[a_1, b_1] \cdot [a_2, b_2] \cdots [a_n, b_n] &= 1, \\
[c, d] &= 1.
\end{align*}
\]
Building Blocks

Luttinger surgeries on product manifolds \( \Sigma_n \times \Sigma_2 \)

\[ Y_n(1/p, 1/q) \text{ is minimal} \]

Since all the torus surgeries listed above are Luttinger surgeries when \( m = 1 \) and the Luttinger surgery preserves minimality [T.-J Li, Chung-I Ho], \( Y_n(1/p, 1/q) \) is a minimal symplectic 4-manifold.
$Y_n(1/p, 1/q)$ is minimal

Since all the torus surgeries listed above are Luttinger surgeries when $m = 1$ and the Luttinger surgery preserves minimality [T.-J Li, Chung-I Ho], $Y_n(1/p, 1/q)$ is a minimal symplectic 4-manifold.
Construction of the exotic

\[ M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}^2} \]
Theorem

Let $M$ be $M = (2n + 2k - 3)\mathbb{CP}^2 \#(6n + 2k - 3)\overline{\mathbb{CP}^2}$ for any $n \geq 1$, $k \geq 1$, and $(n, k) \neq (1, 1)$.

There exists a new family of smooth, closed, simply-connected, minimal, symplectic 4-manifolds and an infinite family of non-symplectic 4-manifolds that is homeomorphic but not diffeomorphic to $M$ that can obtained by the sequence of Luttinger surgeries and a single generalized torus surgery on Lefschetz fibrations.
Construction of the exotic $M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}}^2$ $X(n, k)$

first building block: $Y(n, k) = \Sigma_k \times S^2 \# 4n\mathbb{CP}^2$

Recall that $Y(n, k)$ has a genus $2k + n - 1$ symplectic submanifold $\Sigma_{2k+n-1} \subset Y(n, k)$, which is a regular fiber of the Lefschetz fibration.

$Y(n, k) = \Sigma_k \times S^2 \# 4n\mathbb{CP}^2$ is endowed with the symplectic structure induced from the given Lefschetz fibration.

second building block: 4-manifold $Y_{2k+n-1}(1, 1)$

The other building block will be the symplectic 4-manifold $Y_{2k+n-1}(1, 1)$ along the symplectic submanifold $\Sigma_{2k+n-1}'$, where we set $g = 2k + n - 1$ and $p = q = m = 1$.

$X(n, k)$

$X(n, k)$ denotes the symplectic 4-manifold obtained by the symplectic fiber sum of $Y(n, k)$ and $Y_{2k+n-1}(1, 1)$ along the surfaces $\Sigma_{2k+n-1}$ and $\Sigma_{2k+n-1}'$. 
Construction of the exotic \( M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}}^2 \)

First building block: \( Y(n, k) = \Sigma_k \times S^2 \# 4n\mathbb{CP}^2 \)

Recall that \( Y(n, k) \) has a genus \( 2k + n - 1 \) symplectic submanifold \( \Sigma_{2k+n-1} \subset Y(n, k) \), which is a regular fiber of the Lefschetz fibration.

\( Y(n, k) = \Sigma_k \times S^2 \# 4n\mathbb{CP}^2 \) is endowed with the symplectic structure induced from the given Lefschetz fibration.

Second building block: 4-manifold \( Y_{2k+n-1}(1, 1) \)

The other building block will be the symplectic 4-manifold \( Y_{2k+n-1}(1, 1) \) along the symplectic submanifold \( \Sigma'_{2k+n-1} \), where we set \( g = 2k + n - 1 \) and \( p = q = m = 1 \).

\( X(n, k) \)

\( X(n, k) \) denotes the symplectic 4-manifold obtained by the symplectic fiber sum of \( Y(n, k) \) and \( Y_{2k+n-1}(1, 1) \) along the surfaces \( \Sigma_{2k+n-1} \) and \( \Sigma'_{2k+n-1} \).
Construction of the exotic $M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}}^2$ $X(n, k)$

**first building block: $Y(n, k) = \Sigma_k \times S^2 \# 4n\overline{\mathbb{CP}}^2$**

Recall that $Y(n, k)$ has a genus $2k + n - 1$ symplectic submanifold $\Sigma_{2k+n-1} \subset Y(n, k)$, which is a regular fiber of the Lefschetz fibration. $Y(n, k) = \Sigma_k \times S^2 \# 4n\overline{\mathbb{CP}}^2$ is endowed with the symplectic structure induced from the given Lefschetz fibration.

**second building block: 4-manifold $Y_{2k+n-1}(1, 1)$**

The other building block will be the symplectic 4-manifold $Y_{2k+n-1}(1, 1)$ along the symplectic submanifold $\Sigma'_{2k+n-1}$, where we set $g = 2k + n - 1$ and $p = q = m = 1$.

$X(n, k)$

$X(n, k)$ denotes the symplectic 4-manifold obtained by the symplectic fiber sum of $Y(n, k)$ and $Y_{2k+n-1}(1, 1)$ along the surfaces $\Sigma_{2k+n-1}$ and $\Sigma'_{2k+n-1}$. 
Construction of the exotic manifold \( M = (2n + 2k - 3) \mathbb{CP}^2 \# (6n + 2k - 3) \overline{\mathbb{CP}}^2 \)

\( X(n, k) \)

first building block: \( Y(n, k) = \Sigma_k \times S^2 \# 4n \mathbb{CP}^2 \)

Recall that \( Y(n, k) \) has a genus \( 2k + n - 1 \) symplectic submanifold \( \Sigma_{2k+n-1} \subset Y(n, k) \), which is a regular fiber of the Lefschetz fibration.

\( Y(n, k) = \Sigma_k \times S^2 \# 4n \overline{\mathbb{CP}^2} \) is endowed with the symplectic structure induced from the given Lefschetz fibration.

second building block: 4-manifold \( Y_{2k+n-1}(1, 1) \)

The other building block will be the symplectic 4-manifold \( Y_{2k+n-1}(1, 1) \) along the symplectic submanifold \( \Sigma'_{2k+n-1} \), where we set \( g = 2k + n - 1 \) and \( p = q = m = 1 \).

\( X(n, k) \)

\( X(n, k) \) denotes the symplectic 4-manifold obtained by the symplectic fiber sum of \( Y(n, k) \) and \( Y_{2k+n-1}(1, 1) \) along the surfaces \( \Sigma_{2k+n-1} \) and \( \Sigma'_{2k+n-1} \).
first building block: $Y(n, k) = \Sigma_k \times S^2 \# 4n\overline{CP}^2$

Recall that $Y(n, k)$ has a genus $2k + n - 1$ symplectic submanifold $\Sigma_{2k+n-1} \subset Y(n, k)$, which is a regular fiber of the Lefschetz fibration.

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second building block: 4-manifold $Y_{2k+n-1}(1, 1)$

The other building block will be the symplectic 4-manifold $Y_{2k+n-1}(1, 1)$ along the symplectic submanifold $\Sigma'_{2k+n-1}$, where we set $g = 2k + n - 1$ and $p = q = m = 1$.

$X(n, k)$

$X(n, k)$ denotes the symplectic 4-manifold obtained by the symplectic fiber sum of $Y(n, k)$ and $Y_{2k+n-1}(1, 1)$ along the surfaces $\Sigma_{2k+n-1}$ and $\Sigma'_{2k+n-1}$. 
Construction of the exotic $X(n, k)$

$$M = (2n + 2k - 3) \mathbb{CP}^2 \# (6n + 2k - 3) \overline{\mathbb{CP}}^2$$

first building block: $Y(n, k) = \Sigma_k \times S^2 \# 4n \overline{\mathbb{CP}}^2$

Recall that $Y(n, k)$ has a genus $2k + n - 1$ symplectic submanifold $\Sigma_{2k+n-1} \subset Y(n, k)$, which is a regular fiber of the Lefschetz fibration.

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The other building block will be the symplectic 4-manifold $Y_{2k+n-1}(1, 1)$ along the symplectic submanifold $\Sigma'_{2k+n-1}$, where we set $g = 2k + n - 1$ and $p = q = m = 1$.

$X(n, k)$

$X(n, k)$ denotes the symplectic 4-manifold obtained by the symplectic fiber sum of $Y(n, k)$ and $Y_{2k+n-1}(1, 1)$ along the surfaces $\Sigma_{2k+n-1}$ and $\Sigma'_{2k+n-1}$. 
Lemma

$X(n, k)$ is symplectic.

Proof

It follows from Gompf’s theorem. If $X$, $Y$ are symplectic manifolds and $\Sigma$ is a symplectic submanifold then according to theorem of Gompf $X \#_\psi Y$ admits a symplectic structure.
Lemma

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Proof

It follows from Gompf’s theorem.

If $X$, $Y$ are symplectic manifolds and $\Sigma$ is a symplectic submanifold then according to theorem of Gompf $X \# \psi Y$ admits a symplectic structure.
gluing diffeomorphism $\psi : \Sigma_{2k+n-1} \times S^1 \longrightarrow \Sigma'_g \times S^1$

We choose an orientation-reversing gluing diffeomorphism
$\psi : \Sigma_{2k+n-1} \times S^1 \longrightarrow \Sigma'_g \times S^1$
that maps the generators of the fundamental groups as follows:

$\psi_*(a_1) = a'_1, \psi_*(b_1) = b'_1, \psi_*(a_2) = a'_2, \psi_*(b_2) = b'_2, \cdots,$
$\psi_*(a_{2k}) = a'_{2k}, \psi_*(b_{2k}) = b'_{2k}, \psi_*(e_1) = a'_{2k+1}, \psi_*(e_2) = b'_{2k+1}, \cdots,$
$\psi_*(e_{2n-3}) = a'_{2k+n-1}, \psi_*(e_{2n-2}) = b'_{2k+n-1}, \psi_*(\mu) = (\mu')^{-1}.$

$\pi_1(\Sigma_{2k+n-1} \times S^1)$

Recall that loops $a_1, b_1, \ldots, a_{2k}, b_{2k}$ generate the inclusion-induced image of
$\pi_1(\Sigma_{2k+n-1} \times S^1)$ inside $\pi_1(Y(n,k) \setminus \nu \Sigma_{2k+n-1}).$
This is due to the fact that the loops $e_1, e_2, \ldots, e_{2n-2}, e_{2n-1}$ and the normal circle to
$\mu = \{pt\} \times S^1$ to $\Sigma_{2k+n-1}$ are all nullhomotopic in $\pi_1(Y(n,k) \setminus \nu \Sigma_{2k+n-1}).$
As before, let $a'_1, b'_1, \ldots, a'_{2k+n-1}, b'_{2k+n-1}$ and $\mu' = [c,d]$ generate $\pi_1(\Sigma'_{2k+n-1} \times S^1)$ in
$\pi_1(Y_{2k+n-1}(1,1) \setminus \nu \Sigma'_{2k+n-1}).$
Construction of the exotic \( M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\mathbb{CP}^2 \) \( X(n, k) \)

**gluing diffeomorphism** \( \psi : \Sigma_{2k+n-1} \times S^1 \longrightarrow \Sigma'_g \times S^1 \)**

We choose an orientation-reversing gluing diffeomorphism

\[ \psi : \Sigma_{2k+n-1} \times S^1 \longrightarrow \Sigma'_g \times S^1 \]

that maps the generators of the fundamental groups as follows:

\[
\begin{align*}
\psi_*(a_1) &= a'_1, & \psi_*(b_1) &= b'_1, & \psi_*(a_2) &= a'_2, & \psi_*(b_2) &= b'_2, & \cdots, \\
\psi_*(a_{2k}) &= a'_{2k}, & \psi_*(b_{2k}) &= b'_{2k}, & \psi_*(e_1) &= a'_{2k+1}, & \psi_*(e_2) &= b'_{2k+1}, & \cdots, \\
\psi_*(e_{2n-3}) &= a'_{2k+n-1}, & \psi_*(e_{2n-2}) &= b'_{2k+n-1}, & \psi_*(\mu) &= (\mu')^{-1}.
\end{align*}
\]

\[ \pi_1(\Sigma_{2k+n-1} \times S^1) \]

Recall that loops \( a_1, b_1, \ldots, a_{2k}, \) and \( b_{2k} \) generate the inclusion-induced image of \( \pi_1(\Sigma_{2k+n-1} \times S^1) \) inside \( \pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1}) \).

This is due to the fact that the loops \( e_1, e_2, \ldots, e_{2n-2}, e_{2n-1} \) and the normal circle to \( \mu = \{pt\} \times S^1 \) to \( \Sigma_{2k+n-1} \) are all nullhomotopic in \( \pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1}) \).

As before, let \( a'_1, b'_1, \ldots, a'_{2k+n-1}, b'_{2k+n-1} \) and \( \mu' = [c, d] \) generate \( \pi_1(\Sigma'_{2k+n-1} \times S^1) \) in \( \pi_1(Y_{2k+n-1}(1, 1) \setminus \nu \Sigma'_{2k+n-1}) \).
gluing diffeomorphism $\psi : \Sigma_{2k+n-1} \times S^1 \longrightarrow \Sigma_g \times S^1$

We choose an orientation-reversing gluing diffeomorphism $\psi : \Sigma_{2k+n-1} \times S^1 \longrightarrow \Sigma_g \times S^1$ that maps the generators of the fundamental groups as follows:

$\psi(a_1) = a_1', \psi(b_1) = b_1', \psi(a_2) = a_2', \psi(b_2) = b_2', \cdots$, 
$\psi(a_{2k}) = a_{2k}', \psi(b_{2k}) = b_{2k}', \psi(e_1) = a_{2k+1}', \psi(e_2) = b_{2k+1}', \cdots$, 
$\psi(e_{2n-3}) = a_{2k+n-1}', \psi(e_{2n-2}) = b_{2k+n-1}', \psi(\mu) = (\mu')^{-1}$.

$\pi_1(\Sigma_{2k+n-1} \times S^1)$

Recall that loops $a_1, b_1, \ldots, a_{2k}$, and $b_{2k}$ generate the inclusion-induced image of $\pi_1(\Sigma_{2k+n-1} \times S^1)$ inside $\pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1})$.

This is due to the fact that the loops $e_1, e_2, \ldots, e_{2n-2}, e_{2n-1}$ and the normal circle to $\mu = \{pt\} \times S^1$ to $\Sigma_{2k+n-1}$ are all nullhomotopic in $\pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1})$.

As before, let $a_1', b_1', \ldots, a_{2k+n-1}', b_{2k+n-1}'$ and $\mu' = [c, d]$ generate $\pi_1(\Sigma_{2k+n-1}' \times S^1)$ in $\pi_1(Y_{2k+n-1}(1, 1) \setminus \nu \Sigma_{2k+n-1}')$.
We choose an orientation-reversing gluing diffeomorphism that maps the generators of the fundamental groups as follows:

\[
\begin{align*}
\psi_*(a_1) &= a'_1, \quad \psi_*(b_1) = b'_1, \quad \psi_*(a_2) = a'_2, \quad \psi_*(b_2) = b'_2, \quad \cdots, \\
\psi_*(a_{2k}) &= a'_{2k}, \quad \psi_*(b_{2k}) = b'_{2k}, \quad \psi_*(e_1) = a'_{2k+1}, \quad \psi_*(e_2) = b'_{2k+1}, \quad \cdots, \\
\psi_*(e_{2n-3}) &= a'_{2k+n-1}, \quad \psi_*(e_{2n-2}) = b'_{2k+n-1}, \quad \psi_*(\mu) = (\mu')^{-1}.
\end{align*}
\]

Recall that loops \(a_1, b_1, \cdots, a_{2k}, b_{2k}\) generate the inclusion-induced image of \(\pi_1(\Sigma_{2k+n-1} \times S^1)\) inside \(\pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1})\).

This is due to the fact that the loops \(e_1, e_2, \cdots, e_{2n-2}, e_{2n-1}\) and the normal circle to \(\mu = \{pt\} \times S^1\) to \(\Sigma_{2k+n-1}\) are all nullhomotopic in \(\pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1})\).

As before, let \(a'_1, b'_1, \cdots, a'_{2k+n-1}, b'_{2k+n-1}\) and \(\mu' = [c, d]\) generate \(\pi_1(\Sigma'_{2k+n-1} \times S^1)\) in \(\pi_1(Y_{2k+n-1}(1, 1) \setminus \nu \Sigma'_{2k+n-1})\).
gluing diffeomorphism $\psi : \Sigma_{2k+n-1} \times S^1 \rightarrow \Sigma_g \times S^1$

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$\psi_*(a_1) = a'_1, \psi_*(b_1) = b'_1, \psi_*(a_2) = a'_2, \psi_*(b_2) = b'_2, \cdots,$

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$\pi_1(\Sigma_{2k+n-1} \times S^1)$

Recall that loops $a_1, b_1, \cdots, a_{2k}$, and $b_{2k}$ generate the inclusion-induced image of $\pi_1(\Sigma_{2k+n-1} \times S^1)$ inside $\pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1})$.

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As before, let $a'_1, b'_1, \cdots, a'_{2k+n-1}, b'_{2k+n-1}$ and $\mu' = [c, d]$ generate $\pi_1(\Sigma'_{2k+n-1} \times S^1)$ in $\pi_1(Y_{2k+n-1}(1, 1) \setminus \nu \Sigma'_{2k+n-1})$. 
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We choose an orientation-reversing gluing diffeomorphism $\psi : \Sigma_{2k+n-1} \times S^1 \to \Sigma'_g \times S^1$ that maps the generators of the fundamental groups as follows:

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$\psi \ast (e_{2n-3}) = a'_{2k+n-1}, \psi \ast (e_{2n-2}) = b'_{2k+n-1}, \psi \ast (\mu) = (\mu')^{-1}.$

$\pi_1(\Sigma_{2k+n-1} \times S^1)$

Recall that loops $a_1, b_1, \cdots, a_{2k}$ and $b_{2k}$ generate the inclusion-induced image of $\pi_1(\Sigma_{2k+n-1} \times S^1)$ inside $\pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1}).$

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\[ \psi_*(a_1) = a'_1, \psi_*(b_1) = b'_1, \psi_*(a_2) = a'_2, \psi_*(b_2) = b'_2, \ldots, \]

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\[ \psi_*(e_{2n-3}) = a'_{2k+n-1}, \psi_*(e_{2n-2}) = b'_{2k+n-1}, \psi_*(\mu) = (\mu')^{-1}. \]

Recall that loops \( a_1, b_1, \ldots, a_{2k}, \) and \( b_{2k} \) generate the inclusion-induced image of \( \pi_1(\Sigma_{2k+n-1} \times S^1) \) inside \( \pi_1(Y(n,k) \setminus \nu \Sigma_{2k+n-1}) \).

This is due to the fact that the loops \( e_1, e_2, \ldots, e_{2n-2}, e_{2n-1} \) and the normal circle to \( \mu = \{pt\} \times S^1 \) to \( \Sigma_{2k+n-1} \) are all nullhomotopic in \( \pi_1(Y(n,k) \setminus \nu \Sigma_{2k+n-1}) \).

As before, let \( a'_1, b'_1, \ldots, a'_{2k+n-1}, b'_{2k+n-1} \) and \( \mu' = [c, d] \) generate \( \pi_1(\Sigma'_{2k+n-1} \times S^1) \) in \( \pi_1(Y_{2k+n-1}(1,1) \setminus \nu \Sigma'_{2k+n-1}) \).
Construction of the exotic \( M = (2n + 2k - 3)CP^2 \# (6n + 2k - 3)CP^2 \) \( X(n, k) \)

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We choose an orientation-reversing gluing diffeomorphism \( \psi : \Sigma_{2k+n-1} \times \mathbb{S}^1 \longrightarrow \Sigma'_g \times \mathbb{S}^1 \) that maps the generators of the fundamental groups as follows:

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\begin{align*}
\psi_*(a_1) &= a'_1, \\
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&\cdots, \\
\psi_*(a_{2k}) &= a'_{2k}, \\
\psi_*(b_{2k}) &= b'_{2k}, \\
\psi_*(e_{1}) &= a'_{2k+1}, \\
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&\cdots, \\
\psi_*(e_{2n-3}) &= a'_{2k+n-1}, \\
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\end{align*}
\]

**\( \pi_1(\Sigma_{2k+n-1} \times \mathbb{S}^1) \)**

Recall that loops \( a_1, b_1, \cdots, a_{2k}, b_{2k} \) generate the inclusion-induced image of \( \pi_1(\Sigma_{2k+n-1} \times \mathbb{S}^1) \) inside \( \pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1}) \).

This is due to the fact that the loops \( e_1, e_2, \cdots, e_{2n-2}, e_{2n-1} \) and the normal circle to \( \mu = \{pt\} \times \mathbb{S}^1 \) to \( \Sigma_{2k+n-1} \) are all nullhomotopic in \( \pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1}) \).

As before, let \( a'_1, b'_1, \cdots, a'_{2k+n-1}, b'_{2k+n-1} \) and \( \mu' = [c, d] \) generate \( \pi_1(\Sigma'_{2k+n-1} \times \mathbb{S}^1) \) in \( \pi_1(Y_{2k+n-1}(1, 1) \setminus \nu \Sigma'_{2k+n-1}) \).
Construction of the exotic $M = (2n + 2k - 3)CP^2 \# (6n + 2k - 3)CP^2$ $X(n,k)$

Lemma

$X(n,k)$ is simply-connected.

Proof

By the Seifert-Van Kampen theorem, we see that

$$\pi_1(X(n,k)) = \pi_1(Y(n,k) \setminus \nu \Sigma_{2k+n-1}) \ast \pi_1(Y_g(1,1) \setminus \nu \Sigma'_g)$$

$$\langle a_1 = a'_1, b_1 = b'_1, \ldots, a_{2k} = a'_{2k}, b_{2k} = b'_{2k}, e_1 = a_{2k+1}', e_2 = a_{2k+1}', \ldots, e_{2n-1} = a_{2k+n-1}', e_{2n-2} = b_{2k+n-1}', \mu = \mu' = 1 \rangle$$

Since the loops $e_1, e_2, \ldots, e_{2n-3}, e_{2n-2}$, corresponding to the vanishing cycles, and the normal circle to $\mu = \{pt\} \times S^1$ are all nullhomotopic in $Y(n,k) \setminus \nu \Sigma_{2k+n-1}$, we get the following presentation for the fundamental group of $X(n,k)$. 

Nur Saglam (University of Minnesota, School of Mathematics) October 23, 2014 49 / 62
Lemma

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Proof

By the Seifert-Van Kampen theorem, we see that

$$
\pi_1(X(n, k)) = \frac{\pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1}) \ast \pi_1(Y_g(1, 1) \setminus \nu \Sigma'_{g})}{\langle a_1 = a_1', b_1 = b_1', \ldots, a_{2k} = a_{2k}', b_{2k} = b_{2k}', e_1 = a_{2k+1}', e_2 = a_{2k+1}' \ldots, e_{2n-1} = a_{2k+n-1}', e_{2n-2} = b_{2k+n-1}', \mu = \mu' = 1 \rangle}
$$

Since the loops $e_1, e_2, \ldots, e_{2n-3}, e_{2n-2}$, corresponding to the vanishing cycles, and the normal circle to $\mu = \{pt\} \times S^1$ are all nullhomotopic in $Y(n, k) \setminus \nu \Sigma_{2k+n-1}$, we get the following presentation for the fundamental group of $X(n, k)$. 
Lemma

X(n, k) is simply-connected.

Proof

By the Seifert-Van Kampen theorem, we see that

\[ \pi_1(X(n, k)) = \pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1}) \ast \pi_1(Y_g(1, 1) \setminus \nu \Sigma'_{g}) \]

\[ \langle a_1 = a'_1, b_1 = b'_1, \cdots, a_{2k} = a'_{2k}, b_{2k} = b'_{2k}, e_1 = a'_{2k+1}, e_2 = a'_{2k+1}, \cdots, e_{2n-1} = a'_{2k+n-1}, e_{2n-2} = b'_{2k+n-1}, \mu = \mu' = 1 \rangle \]

Since the loops \( e_1, e_2, \cdots, e_{2n-3}, e_{2n-2} \), corresponding to the vanishing cycles, and the normal circle to \( \mu = \{pt\} \times S^1 \) are all nullhomotopic in \( Y(n, k) \setminus \nu \Sigma_{2k+n-1} \), we get the following presentation for the fundamental group of \( X(n, k) \).
Construction of the exotic $M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\mathbb{CP}^2$ $X(n, k)$

Lemma

$X(n, k)$ is simply-connected.

Proof

By the Seifert-Van Kampen theorem, we see that

$$
\pi_1(X(n, k)) = \pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1}) \ast \pi_1(Y_g(1, 1) \setminus \nu' \Sigma'_g)
$$

$$
\langle a_1'=a_1', b_1'=b_1', \cdots a_{2k}'=a_{2k}', b_{2k}'=b_{2k}', e_1'=a_{2k+1}', e_{2n-1}=a_{2k+n-1}', e_{2n-2}=b_{2k+n-1}', \mu=\mu'=1 \rangle
$$

Since the loops $e_1, e_2, \cdots, e_{2n-3}, e_{2n-2}$, corresponding to the vanishing cycles, and the normal circle to $\mu = \{pt\} \times S^1$ are all nullhomotopic in $Y(n, k) \setminus \nu \Sigma_{2k+n-1}$, we get the following presentation for the fundamental group of $X(n, k)$. 
$M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\mathbb{CP}^2$

$\pi_1(X(n, k)) = \langle a_1, b_1, \cdots, a_{2k}, b_{2k}; c, d; |$
\[ \pi_1(X(n, k)) = \langle a_1, b_1, \cdots, a_{2k}, b_{2k}; c, d; | 
\]
\[ [b_1^{-1}, d^{-1}] = a_1, \quad [a_1^{-1}, d] = b_1, \quad [a_{2k}, d] = 1, \]
\[ \cdots \]
\[ [b_{2k}^{-1}, d^{-1}] = a_{2k}, \quad [a_{2k}^{-1}, d] = b_{2k}, \quad [c^{-1}, b_{2k}]^{-1} = d, \]
\[ a_1 b_1 \cdots b_{2k} = 1, \quad b_1 b_2 \cdots b_{2k-1} = [a_{2k}, b_{2k}], \]
\[ [c, d] = 1, \quad [c, b_1] = [a_1, b_1] = 1, \quad [a_1, b_1][a_2, b_2] \cdots [a_{n}, b_{n}] = 1, \]
\[ [a_1, c] = 1, \quad [b_1, c] = 1, \quad a_1 a_{2k} = 1, \]
\[ \cdots \]
\[ [a_{2k}, c] = 1, \quad [b_{2k}, c] = 1, \quad a_{k} a_{k+1} = 1 \rangle. \]
\( \pi_1(X(n, k)) = \langle a_1, b_1, \cdots, a_{2k}, b_{2k}; c, d; | \\n\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
\cdots \\
[b_{2k}^{-1}, d^{-1}] &= a_{2k}, \\
[a_1^{-1}, d] &= b_1, \\
\cdots \\
[a_{2k}^{-1}, d] &= b_{2k}, \\
[a_2k, d] &= 1, \\
[c^{-1}, b_{2k}^{-1}]^{-1} &= d, \\
[d^{-1}, b_{2k}^{-1}] &= c,
\end{align*} \)
\( \pi_1(X(n,k)) = \langle a_1, b_1, \cdots, a_{2k}, b_{2k}; c, d; \mid \]

\[
\begin{align*}
&[b_1^{-1}, d^{-1}] = a_1, \\
&\cdots \\
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&\cdots \\
&[a_{2k}^{-1}, d] = b_{2k}, \\
&b_1 b_2 \cdots b_{2k} = 1, \\
&a_1 b_1 [a_2, b_2] \cdots [a_n, b_n] = 1, \\
&[c, d] = 1, \\
&a_1 b_1 [a_2, b_2] \cdots [a_k, b_k] = 1, \\
&b_{i+1} b_{i+2} \cdots b_{2k-i} = [a_{2k-i+1}, b_{2k-i+1}] \cdots [a_{2k-1}, b_{2k-1}] [a_{2k}, b_{2k}], \\
&[a_1, c] = 1, \\
&\cdots \\
&[a_k, c] = 1, \\
&[b_{2k}, c] = 1, \\
&a_1 a_{2k} = 1, \\
&\cdots \\
&a_k a_{k+1} = 1 \rangle 
\]
\[ \pi_1(X(n, k)) = \langle a_1, b_1, \ldots, a_{2k}, b_{2k}; c, d; \rangle \]

\[
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
\quad \cdots \quad & \\
[b_{2k}^{-1}, d^{-1}] &= a_{2k}, \\
[a_1^{-1}, d] &= b_1, \\
\quad \cdots \quad & \\
[a_{2k}^{-1}, d] &= b_{2k}, \\
\end{align*}
\]

\[
\begin{align*}
[a_1, b_1][a_2, b_2] \cdots [a_n, b_n] &= 1, \\
[c, d] &= 1, \\
[a_1, b_1][a_2, b_2] \cdots [a_k, b_k] &= 1, \\
b_{i+1}b_{i+2} \cdots b_{2k-i} &= [a_{2k-i+1}, b_{2k-i+1}] \cdots [a_{2k-1}, b_{2k-1}][a_{2k}, b_{2k}] \\
[a_1, c] &= 1, \\
\quad \cdots \quad & \\
[a_{2k}, c] &= 1, \\
[b_1, c] &= 1, \\
\quad \cdots \quad & \\
[a_1a_2 \cdots a_{2k}] &= 1, \\
a_{i}a_{i+1} &= 1, \\
\quad \cdots \quad & \\
[a_{k}a_{k+1}] &= 1.
\end{align*}
\]
\[
\pi_1(X(n,k)) = \langle a_1, b_1, \ldots, a_{2k}, b_{2k}; c, d; \rangle \\
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
\ldots \\
[b_{2k}^{-1}, d^{-1}] &= a_{2k}, \\
[a_1^{-1}, d] &= b_1, \\
\ldots \\
[a_{2k}^{-1}, d] &= b_{2k}, \\
[a_2k, d] &= 1, \\
[c^{-1}, b_{2k}]^{-1} &= d, \\
[d^{-1}, b_{2k}^{-1}] &= c,
\end{align*}
\]

\[
\begin{align*}
[a_1, b_1][a_2, b_2] \cdots [a_n, b_n] &= 1, \\
[b_2b_3 \cdots b_{2k-1}] &= [a_{2k}, b_{2k}], \\
\ldots \\
[a_1, b_1][a_2, b_2] \cdots [a_k, b_k] &= 1, \\
[b_{k+1}b_{k+2} \cdots b_{2k-1}] &= [a_{2k-1}, b_{2k-1}], \\
\ldots \\
[a_1, c] &= 1, \\
[b_1, c] &= 1, \\
\ldots \\
a_1a_{2k} &= 1, \\
\ldots \\
a_k a_{k+1} &= 1.
\end{align*}
\]
\[
\pi_1(X(n, k)) = \langle a_1, b_1, \ldots, a_2k, b_{2k}; c, d; \mid [b_1^{-1}, d^{-1}] = a_1, \quad [a_1^{-1}, d] = b_1, \quad [a_{2k}, d] = 1, \\
\ldots, \quad [b_{2k}^{-1}, d^{-1}] = a_{2k}, \quad [a_{2k}^{-1}, d] = b_{2k}, \quad [c^{-1}, b_{2k}]^{-1} = d, \\
\rangle \quad b_1b_2 \cdots b_{2k} = 1, \quad b_2b_3 \cdots b_{2k-1} = [a_{2k}, b_{2k}], \\
\cdots, \quad b_i+1b_{i+2} \cdots b_{2k-i} = \\
\quad [a_{2k-i+1}, b_{2k-i+1}] \cdots [a_{2k-1}, b_{2k-1}] [a_{2k}, b_{2k}] \\
\quad [a_1, c] = 1, \quad [b_1, c] = 1, \quad a_1a_{2k} = 1, \\
\cdots, \quad [a_{2k}, c] = 1, \quad [b_{2k}, c] = 1, \quad a_{k}a_{k+1} = 1 \rangle.
\]
\[
\pi_1(X(n,k)) = \langle a_1, b_1, \cdots, a_{2k}, b_{2k}; c, d; \ |
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
 & \quad \quad \quad \vdots \\
[b_{2k}^{-1}, d^{-1}] &= a_{2k}, \\
[a_1^{-1}, d] &= b_1, \\
 & \quad \quad \quad \vdots \\
[a_{2k}^{-1}, d] &= b_{2k}, \\
[b_1, b_2 \cdots b_{2k}] &= 1, \\
[b_{2k}] &= 1 \\
[a_1, b_1][a_2, b_2] \cdots [a_n, b_n] &= 1 \\
[c, d] &= 1, \\
[a_1, b_1][a_2, b_2] \cdots [a_k, b_k] &= 1
\end{align*}
\]

\[
\begin{align*}
[b_1, c] &= 1, \\
 & \quad \quad \quad \vdots \\
[b_{2k}, c] &= 1
\end{align*}
\]

\[
[a_1 a_2 \cdots a_k a_{k+1}] = 1.
\]
\[ \pi_1(X(n, k)) = \langle a_1, b_1, \ldots, a_{2k}, b_{2k}; c, d; \rangle \]

\[
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
\cdots & \\
[b_{2k}^{-1}, d^{-1}] &= a_{2k}, \\
[a_1^{-1}, d] &= b_1, \\
\cdots & \\
[a_{2k}^{-1}, d] &= b_{2k}, \\
b_1b_2\cdots b_{2k} &= 1, \\
[b_2^{-1}b_3\cdots b_{2k-1}] &= [a_{2k}, b_{2k}], \\
\cdots & \\
[b_{i+1}b_{i+2}\cdots b_{2k-i}] &= [a_{2k-i+1}, b_{2k-i+1}][a_{2k-1}, b_{2k-1}][a_{2k}, b_{2k}] \\
[a_1, c] &= 1, \\
\cdots & \\
[a_{2k}, c] &= 1.
\end{align*}
\]

\[ a_1a_2\cdots a_{2k} = 1, \]

\[ a_1a_{2k+1} = 1. \]
\[ \pi_1(X(n, k)) = \langle a_1, b_1, \ldots, a_{2k}, b_{2k}; c, d; \rangle \]

\[
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
& \quad \cdots \\
[b_{2k}^{-1}, d^{-1}] &= a_{2k}, \\
[a_1^{-1}, d] &= b_1, \\
& \quad \cdots \\
[a_{2k}^{-1}, d] &= b_{2k}, \\
\end{align*}
\]

\[
\begin{align*}
[a_1, b_1][a_2, b_2] \cdots [a_n, b_n] &= 1, \\
[c, d] &= 1, \\
[a_1, b_1][a_2, b_2] \cdots [a_k, b_k] &= 1 \\
[a_1, c] &= 1, \\
& \quad \cdots \\
[a_{2k}, c] &= 1, \\
[b_1, c] &= 1, \\
& \quad \cdots \\
[b_{2k}, c] &= 1, \\
[a_{2k}, d] &= 1, \\
& \quad \cdots \\
a_1a_{2k} &= 1, \\
& \quad \cdots \\
a_ka_{k+1} &= 1.
\end{align*}
\]
Construction of the exotic $M = (2n + 2k - 3)\mathbb{C}P^2 \#(6n + 2k - 3)\mathbb{C}P^2$ $X(n, k)$

$$\pi_1(X(n, k)) = \langle a_1, b_1, \ldots, a_{2k}, b_{2k}; c, d; |$$

$$[b_1^{-1}, d^{-1}] = a_1,$$

$$\ldots$$

$$[b_{2k}^{-1}, d^{-1}] = a_{2k},$$

$$[a_1^{-1}, d] = b_1,$$

$$\ldots$$

$$[a_{2k}^{-1}, d] = b_{2k},$$

$$[a_{2k}, d] = 1,$$

$$[c^{-1}, b_{2k}]^{-1} = d,$$

$$[d^{-1}, b_{2k}^{-1}] = c,$$

$$b_1 b_2 \cdots b_{2k} = 1,$$

$$b_2 b_3 \cdots b_{2k-1} = [a_{2k}, b_{2k}],$$

$$\ldots$$

$$b_{i+1} b_{i+2} \cdots b_{2k-i} =$$

$$[a_{2k-i+1}, b_{2k-i+1}] \cdots [a_{2k-1}, b_{2k-1}] [a_{2k}, b_{2k}]$$

$$[a_1, c] = 1,$$

$$\ldots$$

$$[a_{2k}, c] = 1,$$

$$[b_1, c] = 1,$$

$$\ldots$$

$$[b_{2k}, c] = 1,$$

$$\ldots$$

$$[a_1 a_{2k} = 1,$$

$$\ldots$$

$$a_k a_{k+1} = 1 \rangle.$$
\[ \pi_1(X(n, k)) = \langle a_1, b_1, \ldots, a_{2k}, b_{2k}; c, d; \rangle \]

\[
\begin{align*}
[b_1^{-1}, d^{-1}] &= a_1, \\
\cdots \\
[b_{2k}^{-1}, d^{-1}] &= a_{2k}, \\
[a_1^{-1}, d] &= b_1, \\
\cdots \\
[a_{2k}^{-1}, d] &= b_{2k}, \\
[a_2k, d] &= 1, \\
[c^{-1}, b_{2k}]^{-1} &= d, \\
[d^{-1}, b_{2k}^{-1}] &= c,
\end{align*}
\]

\[
\begin{align*}
[a_1, b_1][a_2, b_2]\cdots[a_n, b_n] &= 1, \\
[c, d] &= 1, \\
[a_1, b_1][a_2, b_2]\cdots[a_k, b_k] &= 1, \\
[b_1b_2\cdots b_{2k}] &= 1, \\
[b_2b_3\cdots b_{2k-1}] &= [a_{2k}, b_{2k}], \\
\cdots \\
[b_{i+1}b_{i+2}\cdots b_{k-i}] &= [a_{2k-i+1}, b_{2k-i+1}]\cdots[a_{2k-1}, b_{2k-1}][a_{2k}, b_{2k}], \\
[a_1, c] &= 1, \\
\cdots \\
[a_{2k}, c] &= 1, \\
[b_1, c] &= 1, \\
\cdots \\
[b_{2k}, c] &= 1, \\
[a_1a_{2k}] &= 1, \\
\cdots \\
a_k[a_k+1] &= 1.
\end{align*}
\]
\[ M = (2n + 2k - 3) \mathbb{CP}^2 \# (6n + 2k - 3) \mathbb{CP}^2 \]

\[ \pi_1(X(n, k)) = \langle a_1, b_1, \ldots, a_{2k}, b_{2k}; c, d; | \]

\[ [b_1^{-1}, d^{-1}] = a_1, \]
\[ \ldots \]
\[ [b_{2k}^{-1}, d^{-1}] = a_{2k}, \]

\[ [a_1^{-1}, d] = b_1, \]
\[ \ldots \]
\[ [a_{2k}^{-1}, d] = b_{2k}, \]

\[ b_1 b_2 \cdots b_{2k} = 1, \]
\[ b_2 b_3 \cdots b_{2k-1} = [a_{2k}, b_{2k}], \]
\[ \ldots \]

\[ [a_1, b_1][a_2, b_2] \cdots [a_n, b_n] = 1 \]
\[ [c, d] = 1, \]
\[ [a_1, b_1][a_2, b_2] \cdots [a_k, b_k] = 1 \]

\[ [a_1, c] = 1, \]
\[ \ldots \]
\[ [a_{2k}, c] = 1, \]
\[ [b_1, c] = 1, \]
\[ \ldots \]
\[ [b_{2k}, c] = 1, \]

\[ [a_1 a_2 = 1, \]
\[ \ldots \]
\[ a_{2k} a_{k+1} = 1 \rangle. \]
To prove $\pi_1(X(n,k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1}$.

Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$.

Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$.

which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$.

By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$.

Lastly, using the relation $[d^{-1}, b_{2k}^{-1}] = c$, we show $c = 1$.

We also have $a_2 = \cdot \cdot \cdot = a_{2k-1} = b_2 = \cdot \cdot \cdot = b_{2k-1} = 1$, for any $1 \leq i \leq 2k - 1$.

Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$. 
To prove $\pi_1(X(n, k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1}$. Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$. Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$, which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$.

By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$.

Lastly, using the relation $[d^{-1}, b_{2k}] = c$, we show $c = 1$.

We also have $a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1$, for any $1 \leq i \leq 2k - 1$.

Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$. 
To prove \( \pi_1(X(n, k)) = 1 \), it is enough to prove that \( b_1 = 1 \), which in turn will imply that all other generators are trivial.

Using the last set of identities, we have \( a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1} \).

Let us rewrite the relation \([a_1^{-1}, d] = b_1\) as \([a_{2k}, d]^{-1} = b_1\).

Since \([a_{2k}, d] = 1\), we obtain \( b_1 = 1 \).

So, \([b_1^{-1}, d^{-1}] = a_1\) implies \( a_1 = 1 \). Hence \( a_2 = 1 \), which implies \( b_{2k} = 1 \), since \([a_{2k}, d]^{-1} = b_2\).

By using \([c^{-1}, b_{2k}]^{-1} = d\), we get \( d = 1 \).

Lastly, using the relation \([d^{-1}, b_{2k}]^{-1} = c\), we show \( c = 1 \).

We also have \( a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1 \), for any \( 1 \leq i \leq 2k - 1 \).

Since \([b_i^{-1}, d^{-1}] = a_i\) and \([a_i^{-1}, d] = b_i\).
To prove $\pi_1(X(n, k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1}$.

Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$.

Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$, which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$.

By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$.

Lastly, using the relation $[d^{-1}, b_{2k}] = c$, we show $c = 1$.

We also have $a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1$, for any $1 \leq i \leq 2k - 1$.

Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$. 
To prove $\pi_1(X(n, k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1}$. Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$. Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$.

which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$.

By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$.

Lastly, using the relation $[d^{-1}, b_{2k}^{-1}] = c$, we show $c = 1$.
To prove $\pi_1(X(n,k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1}$.

Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$.
Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$. 

which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$

By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$

Lastly, using the relation $[d^{-1}, b_{2k}^{-1}] = c$, we show $c = 1$

We also have $a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1$, for any $1 \leq i \leq 2k-1$.
Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$. 
To prove $\pi_1(X(n,k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots , a_k^{-1} = a_{k+1}$.

Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$.
Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$.
which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$

By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$
Lastly, using the relation $[d^{-1}, b_{2k}^{-1}] = c$, we show $c = 1$

We also have $a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1$, for any $1 \leq i \leq 2k - 1$,
Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$. 
To prove $\pi_1(X(n,k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1}$.

Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$.

Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$.

which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$

By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$

Lastly, using the relation $[d^{-1}, b_{2k}^{-1}] = c$, we show $c = 1$

We also have $a_1 = \ldots = a_{2k-1} = b_2 = \ldots = b_{2k-1} = 1$, for any $1 \leq i \leq 2k - 1$.

Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$. 
To prove $\pi_1(X(n, k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1}$. Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$. Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$.

which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$.

By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$.

Lastly, using the relation $[d^{-1}, b_{2k}^{-1}] = c$, we show $c = 1$.

We also have $a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1$, for any $1 \leq i \leq 2k - 1$.

Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$. 
To prove $\pi_1(X(n, k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1}$.

Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$.

Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$.

which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$

By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$

Lastly, using the relation $[d^{-1}, b_{2k}^{-1}] = c$, we show $c = 1$

We also have $a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1$, for any $1 \leq i \leq 2k - 1$.

Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$. 
To prove $\pi_1(X(n, k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1}$. Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$.

Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$.

which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$.

By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$.

Lastly, using the relation $[d^{-1}, b_{2k}^{-1}] = c$, we show $c = 1$.

We also have $a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1$, for any $1 \leq i \leq 2k - 1$.

Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$. 
To prove $\pi_1(X(n, k)) = 1$, it is enough to prove that $b_1 = 1$, which in turn will imply that all other generators are trivial.

Using the last set of identities, we have $a_1^{-1} = a_{2k}, \ldots, a_k^{-1} = a_{k+1}$. Let us rewrite the relation $[a_1^{-1}, d] = b_1$ as $[a_{2k}, d]^{-1} = b_1$. Since $[a_{2k}, d] = 1$, we obtain $b_1 = 1$.

So, $[b_1^{-1}, d^{-1}] = a_1$ implies $a_1 = 1$. Hence $a_{2k} = 1$. which implies $b_{2k} = 1$, since $[a_{2k}, d]^{-1} = b_2$. By using $[c^{-1}, b_{2k}]^{-1} = d$, we get $d = 1$. Lastly, using the relation $[d^{-1}, b_{2k}^{-1}] = c$, we show $c = 1$.

We also have $a_2 = \cdots = a_{2k-1} = b_2 = \cdots = b_{2k-1} = 1$, for any $1 \leq i \leq 2k - 1$, Since $[b_i^{-1}, d^{-1}] = a_i$ and $[a_i^{-1}, d] = b_i$.mars
Construction of the exotic $M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}^2}$ $X(n, k)$

**Lemma**

\[
\alpha(X(n, k)) = 6n + 4k - 4 \\
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$X(n, k)$ is homeomorphic to $M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}^2}$

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\sigma(X(n, k)) &= 8n + 4k - 4 \\
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Nur Saglam (University of Minnesota, School of Mathematics) October 23, 2014
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Construction of the exotic manifold

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Construction of the exotic $M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}}^2$ $X(n, k)$

$X(n, k)$ is minimal

Notice that all $4n$ exceptional spheres $E_1, E_2, \cdots, E_{4n-1}$, and $E_{4n}$, which are the sections of the genus $2k+n-1$ L. F. on $Y(n, k) = \Sigma_k \times S^2 \# 4n\overline{\mathbb{CP}}^2$, meet with the fiber $\Sigma = 2n(\Sigma_{2k+n-1} \times \{pt\}) + \{pt\} \times S^2 - E_1 - E_2 - \cdots E_{4n-1} - E_{4n}$.

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**Construction of the exotic**

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Since symplectic minimality implies smooth minimality [T.-J. Li], $X(n, k)$ is smoothly minimal as well.
**Construction of the exotic**

\[ M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}}^2 \]

\[ X(n, k) \]

---

**Proving that** \( X(n, k) \) **is minimal**

Notice that all \( 4n \) exceptional spheres \( E_1, E_2, \ldots, E_{4n-1}, \) and \( E_{4n}, \)
which are the sections of the genus \( 2k + n - 1 \) L. F. on \( Y(n, k) = \Sigma_k \times S^2 \# 4n\overline{\mathbb{CP}}^2, \)
meet with the fiber \( \Sigma = 2n(\Sigma_{2k+n-1} \times \{pt\}) + \{pt\} \times S^2 - E_1 - E_2 - \cdots - E_{4n-1} - E_{4n}. \)

Furthermore, using the adjunction equality, we see that any embedded symplectic \(-1\) sphere in \( Y(n, k) \) has the form \( rS^2 \equiv E_i, \)

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Notice that all \( 4n \) exceptional spheres \( E_1, E_2, \cdots, E_{4n-1}, \) and \( E_{4n} \), which are the sections of the genus \( 2k + n - 1 \) L. F. on \( Y(n, k) = \Sigma_k \times S^2 \# 4n \mathbb{C}P^2 \), meet with the fiber \( \Sigma = 2n(\Sigma_{2k+n-1} \times \{pt\}) + \{pt\} \times S^2 - E_1 - E_2 - \cdots - E_{4n-1} - E_{4n} \).

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$X(n, k, m)$: infinite family of simply connected, non-symplectic, exotic
$M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}^2}$'s

We replace the building block $Y_g(1, 1)$ with $Y_g(1, m)$, where $|m| \geq 2$.

In the presentation of the fundamental group, this amounts replacing a single relation $[c^{-1}, b_n]^{-1} = d$ (corr. to the L. S. $(a'_n \times d', d', m)$),
with $[c^{-1}, b_n]^{-m} = d$.

Note that the new relation has no effect on our proof of $\pi_1(X(n, k)) = 1$.

Hence, the simplification of the presentation for the fundamental group of $X(n, k, m)$
follows the same steps, and thus $\pi_1(X(n, k, m)) = 1$. 
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$X(n, k, m)$: infinite family of simply connected, non-symplectic, exotic $M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\overline{\mathbb{CP}}^2$'s

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Hence, the simplification of the presentation for the fundamental group of \(X(n, k, m)\) follows the same steps, and thus \(\pi_1(X(n, k, m)) = 1\).
$X(n, k, m)$ is nonsymplectic for any $m \geq 2$

Using the argument similar as in "Constructing infinitely many smooth structures on small 4-manifolds" ([A. Akhmedov, R.I. Baykur, and B.D.Park]) (Section 4), we see that $X(n, k)$ has one basic class up to sign, the canonical class $\pm K_{X(n,k)}$.

Let $X(n, k)_0$ denote the symplectic 4-manifold obtained by performing the following Luttinger surgery on:

$$(a''_n \times d', d', 0/1).$$

It is easy to see that $\pi_1(X(n, k)_0) = \mathbb{Z}$ and the canonical class $K_{X(n,k)_0} = 2[\Sigma_l] + \sum_{j=1}^{4k}[R_j]$, where $R_j$ are rim tori of self-intersection $-1$.

Furthermore, the basic class $\beta_{n,k,m}$ of $X(n, k, m)$ corresponding to the canonical class $K_{X(n,k)_0}$ satisfies

$$SW_{X(n,k,m)}(\beta_{n,k,m}) = SW_{X(n,k)}(K_{X(n,k)}) + (m - 1)SW_{X(n,k)_0}(K_{X(n,k)_0}) = 1 + (m - 1) = m.$$ 

Thus, $X(n, k, m)$ is nonsymplectic for any $m \geq 2$.
Construction of the exotic $M = (2n + 2k - 3)\mathbb{CP}^2 \# (6n + 2k - 3)\mathbb{CP}^2$ $X(n, k, m)$

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Let $X(n, k)_0$ denote the symplectic 4-manifold obtained by performing the following Luttinger surgery on:

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Thus, $X(n, k, m)$ is nonsymplectic for any $m \geq 2$. 
Remarks
symplectic 4-manifolds with the various finitely generated fundamental groups

We choose the gluing diffeomorphism $\psi: \Sigma_{2k+n-1} \times S^1 \rightarrow \Sigma_l \times S^1$ that maps the generators of the fundamental group as follows:

$\psi^*(\alpha_1) = c_1$, $\psi^*(\beta_1) = d_1$, $\psi^*(\alpha_2) = c_2$, $\psi^*(\beta_2) = d_2$, ..., $\psi^*(\alpha_{2k}) = c_{2k}$, $\psi^*(\beta_{2k}) = d_{2k}$,

$\psi^*(\epsilon_1) = c_{2k+1}$, $\psi^*(\epsilon_2) = d_{2k+1}$, ..., $\psi^*(\epsilon_{2n-3}) = c_{2k+n-1}$, $\psi^*(\epsilon_{2n-2}) = d_{2k+n-1}$.

$\psi^*(\mu) = \mu'$

$X(n,k,p,q)$ is symplectic

Follows from Gompf's theorem.
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$\psi_*(\alpha_1) = c_1, \psi_*(\beta_1) = d_1, \psi_*(\alpha_2) = c_2, \psi_*(\beta_2) = d_2, \ldots, \psi_*(\alpha_{2k}) = c_{2k}, \psi_*(\beta_{2k}) = d_{2k},$

$\psi_*(e_1) = c_{2k+1}, \psi_*(e_2) = d_{2k+1}, \ldots, \psi_*(e_{2n-3}) = c_{2k+n-1}, \psi_*(e_{2n-2}) = d_{2k+n-1},$

$\psi_*(\mu) = \mu'$

$X(n, k, \bar{p}, \bar{q})$ is symplectic

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\[
\begin{align*}
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\psi_*(\beta_1) &= d_1, \\
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\psi_*(\beta_2) &= d_2, \\
&\vdots \\
\psi_*(\alpha_{2k}) &= c_{2k}, \\
\psi_*(\beta_{2k}) &= d_{2k}, \\
\psi_*(e_1) &= c_{2k+1}, \\
\psi_*(e_2) &= d_{2k+1}, \\
&\vdots \\
\psi_*(e_{2n-3}) &= c_{2k+n-1}, \\
\psi_*(e_{2n-2}) &= d_{2k+n-1}, \\
\psi_*(\mu) &= \mu'.
\end{align*}
\]

\( X(n, k, \bar{p}, \bar{q}) \) is symplectic

Follows from Gompf's theorem.
By Van Kampen’s theorem, we have

\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \pi_1(Y(n, k) \setminus \nu_{\Sigma_{2k+n-1}}) \ast \pi_1(Y_l(1/p_1, 1/q_1, \ldots, 1/p_l, 1/q_l) \setminus \nu_{\Sigma_l}) \]

\[ \langle \alpha_1 = c_1, \beta_1 = d_1, \alpha_2 = c_2, \beta_2 = d_2, \ldots, \alpha_{2k} = c_{2k}, \beta_{2k} = d_{2k}, e_1 = c_{2k+1}, e_2 = d_{2k+1}, \ldots, e_{2n-3} = c_{2k+n-1}, e_{2n-2} = d_{2k+n-1}, \mu = \mu' \rangle \]

Since the loops \( e_1, e_2, \ldots, e_{2n-3}, e_{2n-2} \) and the normal circle to \( \mu = \{pt\} \times S^1 \) are all nullhomotopic in \( \pi_1(Y(n, k) \setminus \nu_{\Sigma_{2k+n-1}}) \), we get the following presentation for the fundamental group of \( X(n, k, \bar{p}, \bar{q}) \).
By Van Kampen’s theorem, we have

$$\pi_1(X(n,k,\bar{p},\bar{q})) =$$

$$\pi_1(Y(n,k) \setminus \nu \Sigma_{2k+n-1}) \ast \pi_1(Y_{1/p_1,1/q_1,\cdots,1/p_l,1/q_l} \setminus \nu \Sigma_l)$$

$$\langle \alpha_1=c_1, \beta_1=d_1, \alpha_2=c_2, \beta_2=d_2, \cdots, \alpha_{2k}=c_{2k}, \beta_{2k}=d_{2k}, e_1=c_{2k+1}, e_2=d_{2k+1}, \cdots, e_{2n-3}=c_{2k+n-1}, e_{2n-2}=d_{2k+n-1}, \mu=\mu' \rangle$$

Since the loops $e_1, e_2, \cdots, e_{2n-3}, e_{2n-2}$ and the normal circle to $\mu = \{pt\} \times S^1$ are all nullhomotopic in $\pi_1(Y(n,k) \setminus \nu \Sigma_{2k+n-1})$, we get the following presentation for the fundamental group of $X(n,k,\bar{p},\bar{q})$. 
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$$\pi_1(X(n, k, \bar{p}, \bar{q}))$$

By Van Kampen's theorem, we have

$$\pi_1(X(n, k, \bar{p}, \bar{q})) = \pi_1(Y(n, k) \setminus \nu \Sigma_{2k+n-1}) * \pi_1(Y_1(1/p_1, 1/q_1, \cdots, 1/p_l, 1/q_l) \nu \Sigma_l)$$

$$\langle \alpha_1 = c_1, \beta_1 = d_1, \alpha_2 = c_2, \beta_2 = d_2, \cdots, \alpha_{2k} = c_{2k}, \beta_{2k} = d_{2k}, e_1 = c_{2k+1}, e_2 = d_{2k+1}, \cdots, e_{2n-3} = c_{2k+n-1}, e_{2n-2} = d_{2k+n-1}, \mu = \mu' \rangle$$

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$\pi_1(X(n, k, p, q))$

$\pi_1(X(n, k, p, q)) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; |$

\[
\begin{align*}
&\left[ a_1, d_1 \right] = \left[ c_1, d_1 \right] = \left[ d_1, c_2 \right] = \left[ a_1, d_2 \right] = \left[ c_2, d_2 \right] = \left[ a_2, d_3 \right] = \left[ c_3, d_3 \right] = \cdots = \left[ a_1, d_{2k} \right] = \left[ c_{2k}, d_{2k} \right] = \left[ a_{2k}, d_{2k+1} \right] = \left[ c_{2k+1}, d_{2k+1} \right] = \left[ a_{2k+1}, d_{2k+2} \right] = \left[ c_{2k+2}, d_{2k+2} \right] = \cdots = \left[ a_1, d_{2k+2} \right] = \left[ c_{2k+2}, d_{2k+2} \right] = \left[ a_{2k+2}, d_{2k+3} \right] = \left[ c_{2k+3}, d_{2k+3} \right] \rangle
\end{align*}
\]
$$\pi_1(X(n, k, \bar{p}, \bar{q}))$$

$$\pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; |$$

$$[b_1^{-1}, d_1^{-1}] = [c_1^{-1}, b_2] = d_1^{q_1} \quad [a_1, c_1] = 1 \quad [a_2, c_1] = 1$$
$$a_1 [b_2^{-1}, d_2^{-1}] = [c_1^{-1}, b_1] = d_2^{q_2} \quad [a_1, c_2] = 1 \quad [a_2, c_2] = 1$$
$$a_2$$
$$[a_1^{-1}, d_1] = b_1 \quad [d_1^{-1}, b_2^{-1}] = c_1^{p_1} \quad [a_1, d_1] = 1 \quad [a_2, d_1] = 1$$
$$[a_2^{-1}, d_2] = b_2 \quad [d_2^{-1}, b_1^{-1}] = c_2^{p_2} \quad [b_1, c_1] = 1 \quad [b_2, c_2] = 1$$

$$[a_1, b_1][a_2, b_2] = 1, \prod_{j=1}^{2k} [c_j, d_j] = 1$$

$$[a_1^{-1}, d_3] = c_3^{p_3} \quad [a_2^{-1}, c_3^{-1}] = d_3^{p_3} \quad [b_1, c_3] = 1 \quad [b_2, d_3] = 1 \quad c_1 c_{2k} = 1$$
$$\ldots$$
$$\ldots$$

$$[a_1^{-1}, d_{2k}] = c_{2k}^{p_{2k}} \quad [a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{p_{2k}} \quad [b_1, c_{2k}] = 1 \quad [b_2, d_{2k}] = 1 \quad c_k c_{k+1} = 1$$

$$d_1 d_2 \cdots d_{2k} = 1$$
$$d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}]$$
$$\ldots$$
$$d_i+1 d_{i+2} \cdots d_{2k-i} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}] [c_{2k}, d_{2k}]$$
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; \rangle \]

\[ [b_1^{-1}, d_1^{-1}] = \]
\[ a_1 [b_2^{-1}, d_2^{-1}] = \]
\[ a_2 [a_1^{-1}, d_1] = b_1 \]
\[ [a_2^{-1}, d_2] = b_2 \]

\[ [c_1^{-1}, b_2] = d_1^{q_1} \]
\[ [c_1^{-1}, b_1] = d_2^{q_2} \]
\[ [d_1^{-1}, b_2^{-1}] = c_1^{p_1} \]
\[ [d_1^{-1}, b_1^{-1}] = c_2^{p_2} \]

\[ [a_1, b_1][a_2, b_2] = 1, \prod_{j=1}^{2k} [c_j, d_j] = 1 \]

\[ [a_1^{-1}, d_3^{-1}] = d_3^{p_3} \]
\[ [a_2^{-1}, c_3^{-1}] = d_3^{p_3} \]
\[ \cdots \]
\[ [a_1^{-1}, d_{2k}^{-1}] = d_2^{p_2k} \]
\[ [a_2^{-1}, c_{2k}^{-1}] = d_2^{p_2k} \]

\[ [b_1, c_3] = 1 \]
\[ [b_1, c_{2k}] = 1 \]
\[ [b_2, d_3] = 1 \]
\[ [b_2, d_{2k}] = 1 \]
\[ c_1 c_{2k} = 1 \]
\[ c_k c_{k+1} = 1 \]

\[ d_1 d_2 \cdots d_{2k} = 1 \]
\[ d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}] \]
\[ \cdots \]
\[ d_{i+1} d_{i+2} \cdots d_{2k-1} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}][c_{2k}, d_{2k}] \]
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) \]

\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; | \]

\[
\begin{align*}
[b_1^{-1}, d_1^{-1}] = & \quad [c_1^{-1}, b_2] = d_1^{q_1} & [a_1, c_1] = 1 & [a_2, c_1] = 1 \\
[a_1 \ [b_2^{-1}, d_2^{-1}] = & \quad [c_1^{-1}, b_1] = d_2^{q_2} & [a_1, c_2] = 1 & [a_2, c_2] = 1 \\
a_2 & \quad [d_1^{-1}, b_2^{-1}] = c_1^{p_1} & [a_1, d_2] = 1 & [a_2, d_1] = 1 \\
[a_1^{-1}, d_1] = b_1 & \quad [d_2^{-1}, b_1^{-1}] = c_2^{p_2} & [b_1, c_1] = 1 & [b_2, c_2] = 1 \\
[a_2^{-1}, d_2] = b_2 & \\
\end{align*}
\]

\[ [a_1, b_1][a_2, b_2] = 1, \prod_{j=1}^{2k} [c_j, d_j] = 1 \]

\[
\begin{align*}
[a_1^{-1}, d_3^{-1}] = & \quad [a_2^{-1}, c_3^{-1}] = \cdots = d_3^{p_3} & [b_1, c_3] = 1 & [b_2, d_3] = 1 & c_1c_2k = 1 \\
& \quad \cdots \quad \cdots \quad \cdots \\
[a_1^{-1}, d_{2k}^{-1}] = & \quad [a_2^{-1}, c_{2k}^{-1}] = \cdots = d_{2k}^{p_{2k}} & [b_1, c_{2k}] = 1 & [b_2, d_{2k}] = 1 & c_kc_{k+1} = 1 \\
& \quad \cdots \quad \cdots \quad \cdots \\
d_1d_2 \cdots d_{2k} = 1 & \quad [c_2k, d_{2k}] \\
d_2d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}] & \quad \cdots \\
d_{i+1}d_{i+2} \cdots d_{2k-1} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}] [c_{2k}, d_{2k}] \\
\end{align*}
\]
$$\pi_1(X(n, k, \overline{p}, \overline{q}))$$

$$\pi_1(X(n, k, \overline{p}, \overline{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; |$$

$$[b_1^{-1}, d_1^{-1}] = [c_1^{-1}, b_2] = d_1^{q_1}$$
$$a_1 [b_2^{-1}, d_2^{-1}] = [c_1^{-1}, b_1] = d_2^{q_2}$$
$$a_2 [a_1^{-1}, d_1] = b_1$$
$$[a_1^{-1}, d_2] = b_2$$

$$[a_1^{-1}, d_3] = c_2^{p_3}$$
$$[a_2^{-1}, c_3^{-1}] = d_3^{p_3}$$
$$[b_1, c_3] = 1$$
$$[b_2, d_3] = 1$$
$$c_1 c_{2k} = 1$$

$$[a_1^{-1}, d_4] = c_2^{p_4}$$
$$[a_2^{-1}, c_4^{-1}] = d_4^{p_4}$$
$$[b_1, c_4^{-1}] = d_4^{p_4}$$
$$[b_2, d_4] = 1$$
$$c_1 c_{2k} = 1$$

$$\cdots$$

$$[a_1^{-1}, d_{2k}] = c_2^{p_{2k}}$$
$$[a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{p_{2k}}$$
$$[b_1, c_{2k}] = 1$$
$$[b_2, d_{2k}] = 1$$
$$c_k c_{k+1} = 1$$

$$d_1 d_2 \cdots d_{2k} = 1$$
$$d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}]$$
$$\cdots$$
$$d_{i+1} d_{i+2} \cdots d_{2k-i} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}] [c_{2k}, d_{2k}]$$
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; | \]
\[ b_1^{-1}, d_1^{-1} = \]
\[ a_1 [b_2^{-1}, d_2^{-1}] = \]
\[ a_2 [a_1^{-1}, d_1] = b_1 \]
\[ [a_2^{-1}, d_2] = b_2 \]
\[ c_1^{-1}, b_2 = d_1^{q_1} \]
\[ [c_1^{-1}, b_1] = d_2^{q_2} \]
\[ [a_1, c_1] = 1 \]
\[ [a_2, c_1] = 1 \]
\[ [a_1, c_2] = 1 \]
\[ [a_2, c_2] = 1 \]
\[ [a_1, d_1] = 1 \]
\[ [a_2, d_1] = 1 \]
\[ [b_1, c_1] = 1 \]
\[ [b_2, c_1] = 1 \]
\[ [a_1, b_1][a_2, b_2] = 1, \prod_{j=1}^{2k} [c_j, d_j] = 1 \]
\[ [a_1^{-1}, d_3^{-1}] = c_3^{p_1} \]
\[ [a_2^{-1}, c_3^{-1}] = d_3^{p_1} \]
\[ [b_1, c_3] = 1 \]
\[ [b_2, d_3] = 1 \]
\[ c_1 c_{2k} = 1 \]
\[ [a_1^{-1}, d_{2k}^{-1}] = c_{2k}^{p_1} \]
\[ [a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{p_1} \]
\[ [b_1, c_{2k}] = 1 \]
\[ [b_2, d_{2k}] = 1 \]
\[ c_k c_{k+1} = 1 \]
\[ d_1 d_2 \cdots d_{2k} = 1 \]
\[ d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}] \]
\[ \cdots \]
\[ d_{i+1} d_{i+2} \cdots d_{2k-1} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}] [c_{2k}, d_{2k}] \]
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; | \]

\[
\begin{align*}
[b_1^{-1}, d_1^{-1}] = 1 \\
a_1 [b_2^{-1}, d_2^{-1}] = 1 \\
a_2 [a_1^{-1}, d_1] = b_1 \\
a_2 [a_2^{-1}, d_2] = b_2 \\
[c_1^{-1}, b_2] = d_1^{q_1} \\
[c_1^{-1}, b_1] = d_2^{q_2} \\
[d_1^{-1}, b_2^{-1}] = c_1^{p_1} \\
[d_2^{-1}, b_1^{-1}] = c_2^{p_2} \\
[a_1, c_1] = 1 \\
[a_2, c_1] = 1 \\
[a_1, c_2] = 1 \\
[a_2, c_2] = 1 \\
[a_1, d_2] = 1 \\
[a_2, d_1] = 1 \\
[a_1, d_1] = 1 \\
[b_1, c_1] = 1 \\
[b_2, c_2] = 1 \\
[a_1^{-1}, d_3^{-1}] = d_3^{p_3} \\
[a_2^{-1}, c_3^{-1}] = d_3^{p_3} \\
[b_1, c_3] = 1 \\
[b_2, d_3] = 1 \\
[a_1^{-1}, d_{2k}^{-1}] = d_{2k}^{p_{2k}} \\
[a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{p_{2k}} \\
[b_1, c_{2k}] = 1 \\
[b_2, d_{2k}] = 1 \\
\prod_{j=1}^{2k} [c_j, d_j] = 1
\end{align*}
\]

\[
\begin{align*}
d_1 d_2 \cdots d_{2k} = 1 \\
d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}] \\
\cdots \\
d_{2k-2} d_{2k-1} = [c_{2k-1}, d_{2k-1}] \\
d_{2k-3} d_{2k-2} \cdots d_2 = [c_{2k-1}, d_{2k-1}] [c_{2k}, d_{2k}] \\
\end{align*}
\]
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) \]

\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; | \]

\[
\begin{align*}
[b_1^{-1}, d_1^{-1}] &= [c_1^{-1}, b_2] = d_1^{q_1} & [a_1, c_1] = 1 & [a_2, c_1] = 1 \\
[a_1 [b_2^{-1}, d_2^{-1}] &= [c_1^{-1}, b_1] = d_2^{q_2} & [a_1, c_2] = 1 & [a_2, c_2] = 1 \\
a_2 & [d_1^{-1}, b_2^{-1}] = c_1^{p_1} & [a_1, d_2] = 1 & [a_2, d_1] = 1 \\
[a_1^{-1}, d_1] = b_1 & [d_2^{-1}, b_1^{-1}] = c_2^{p_2} & [b_1, c_1] = 1 & [b_2, c_2] = 1 \\
[a_2^{-1}, d_2] = b_2 & \end{align*}
\]

\[ [a_1, b_1][a_2, b_2] = 1, \prod_{j=1}^{2k} [c_j, d_j] = 1 \]

\[
\begin{align*}
[a_1^{-1}, d_3^{-1}] &= d_3^{p_3} & [a_2^{-1}, c_3^{-1}] &= d_3^{p_3} & [b_1, c_3] = 1 & [b_2, d_3] = 1 & c_1 c_{2k} = 1 \\
& \cdots & & \cdots & & \cdots & & \cdots \\
[a_1^{-1}, d_{2k}^{-1}] &= d_{2k}^{p_{2k}} & [a_2^{-1}, c_{2k}^{-1}] &= d_{2k}^{p_{2k}} & [b_1, c_{2k}] &= [b_2, d_{2k}] = 1 & c_k c_{k+1} = 1 \\
& \end{align*}
\]

\[ d_1 d_2 \cdots d_{2k} = 1 \]

\[ d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}] \]

\[ \cdots \]

\[ d_i d_{i+2} \cdots d_{2k-1} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-i}, d_{2k-i}][c_{2k}, d_{2k}] \]
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; \mid \]
\[ [b_1^{-1}, d_1^{-1}] = \]
\[ a_1 [b_2^{-1}, d_2^{-1}] = \]
\[ a_2 \]
\[ [a_1^{-1}, d_1] = b_1 \]
\[ [a_2^{-1}, d_2] = b_2 \]
\[ [c_1^{-1}, b_2] = d_1^{q_1} \quad [a_1, c_1] = 1 \quad [a_2, c_1] = 1 \]
\[ [c_1^{-1}, b_1] = d_2^{q_2} \quad [a_1, c_2] = 1 \quad [a_2, c_2] = 1 \]
\[ [d_1^{-1}, b_2^{-1}] = c_1^{p_1} \quad [a_1, d_2] = 1 \quad [a_2, d_1] = 1 \]
\[ [d_2^{-1}, b_1^{-1}] = c_2^{p_2} \quad [b_1, c_1] = 1 \quad [b_2, c_2] = 1 \]
\[ [a_1, b_1][a_2, b_2] = 1, \quad \Pi_{j=1}^{2k} [c_j, d_j] = 1 \]
\[ [a_1^{-1}, d_3^{-1}] = c_3^{p_3} \quad \cdots \]
\[ [a_2^{-1}, c_3^{-1}] = d_3^{p_3} \quad \cdots \]
\[ [b_1, c_3] = 1 \quad [b_2, d_3] = 1 \quad c_1 c_2 k = 1 \]
\[ \cdots \]
\[ [a_1^{-1}, d_{2k}^{-1}] = c_{2k}^{p_{2k}} \quad \cdots \]
\[ [a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{p_{2k}} \quad \cdots \]
\[ [b_1, c_{2k}] = 1 \quad [b_2, d_{2k}] = 1 \quad c_k c_{k+1} = 1 \]
\[ d_1 d_2 \cdots d_{2k} = 1 \]
\[ d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}] \]
\[ \cdots \]
\[ d_{i+1} d_{i+2} \cdots d_{2k-1} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}] [c_{2k}, d_{2k}] \]
$$\pi_1(X(n, k, \bar{p}, \bar{q}))$$

$$\pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; \rangle$$

$$[b_1^{-1}, d_1^{-1}] = [c_1^{-1}, b_2] = d_1^{q_1}$$
$$a_1 [b_2^{-1}, d_2^{-1}] = [c_1^{-1}, b_1] = d_2^{q_2}$$
$$[a_1, c_1] = 1$$
$$[a_2, c_1] = 1$$
$$[a_2, c_2] = 1$$
$$[a_2, d_1] = 1$$
$$[a_1, d_2] = 1$$
$$[b_2, c_2] = 1$$

$$[a_1, b_1][a_2, b_2] = 1, \quad \prod_{j=1}^{2k} [c_j, d_j] = 1$$

$$[a_1^{-1}, d_1^{-1}] = c_3^{p_3}$$
$$[a_2^{-1}, c_3^{-1}] = d_3^{p_3}$$
$$[b_1, c_3] = 1$$
$$[b_2, d_3] = 1$$
$$c_1 c_{2k} = 1$$

$$\cdots$$

$$[a_1^{-1}, d_{2k}^{-1}] = c_{2k}^{p_{2k}}$$
$$[a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{p_{2k}}$$
$$[b_1, c_{2k}] = 1$$
$$[b_2, d_{2k}] = 1$$
$$c_k c_{k+1} = 1$$

$$d_1 d_2 \cdots d_{2k} = 1$$
$$d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}]$$

$$\cdots$$

$$d_{i+1} d_{i+2} \cdots d_{2k-1} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}][c_{2k}, d_{2k}]$$
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; \rangle \]

\[
\begin{align*}
  [b_1^{-1}, d_1^{-1}] &= [c_1^{-1}, b_2] = d_1^{q_1} \quad [a_1, c_1] = 1 \quad [a_2, c_1] = 1 \\
  a_1 [b_2^{-1}, d_2^{-1}] &= [c_1^{-1}, b_1] = d_2^{q_2} \quad [a_1, c_2] = 1 \quad [a_2, c_2] = 1 \\
  a_2 [a_1^{-1}, d_1] &= b_1 \quad [c_1^{-1}, b_2^{-1}] = c_1^{p_1} \quad [a_1, d_2] = 1 \quad [a_2, d_1] = 1 \\
  [a_2^{-1}, d_2] &= b_2 \quad [d_2^{-1}, b_2^{-1}] = c_2^{p_2} \quad [b_1, c_1] = 1 \quad [b_2, c_2] = 1
\end{align*}
\]

\[
[a_1, b_1][a_2, b_2] = 1, \quad \Pi_{j=1}^{2k} [c_j, d_j] = 1
\]

\[
\begin{align*}
  [a_1^{-1}, d_3^{-1}] &= c_3^{p_3} \quad [a_2^{-1}, c_3^{-1}] = d_3^{p_3} \quad [b_1, c_3] = 1 \quad [b_2, d_3] = 1 \quad c_1 c_{2k} = 1 \\
  \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\
  [a_1^{-1}, d_{2k}^{-1}] &= c_{2k}^{p_{2k}} \quad [a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{p_{2k}} \quad \quad [b_1, c_{2k}] = [b_2, d_{2k}] = 1 \quad c_k c_{k+1} = 1 \\
  \quad [b_1, c_{2k}] &= 1 \\
  \quad [b_2, d_{2k}] &= 1 \\
  &c_k c_{k+1} = 1
\end{align*}
\]

\[
\begin{align*}
  d_1 d_2 \cdots d_{2k} &= 1 \\
  d_2 d_3 \cdots d_{2k-1} &= [c_{2k}, d_{2k}] \\
  \cdots \\
  d_{i+1} d_{i+2} \cdots d_{2k-i} &= [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}] [c_{2k}, d_{2k}]
\end{align*}
\]
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; | \\
\begin{align*}
[b_1^{-1}, d_1^{-1}] &= [c_1^{-1}, b_2] = d_1^{q_1} & [a_1, c_1] = 1 & [a_2, c_1] = 1 \\
[a_1 [b_2^{-1}, d_2^{-1}] &= [c_1^{-1}, b_1] = d_2^{q_2} & [a_1, c_2] = 1 & [a_2, c_2] = 1 \\
a_2 & [d_1^{-1}, b_2^{-1}] = c_1^{p_1} & [a_1, d_1] = 1 & [a_2, d_1] = 1 \\
[a_1^{-1}, d_1] &= b_1 & [b_1, c_1] = 1 & [b_2, c_2] = 1 \\
[a_2^{-1}, d_2] &= b_2 & [b_1, c_2] = 1 & [b_2, c_2] = 1 \\
[a_1, b_1][a_2, b_2] &= 1, \quad \Pi_{j=1}^{2k} [c_j, d_j] = 1 \\
[a_1^{-1}, d_3^{-1}] &= c_3^{p_3} & [a_2^{-1}, c_3^{-1}] &= d_3^{p_3} & [b_1, c_3] = 1 & [b_2, d_3] = 1 & c_1 c_{2k} = 1 \\
\ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots & \quad \ldots \\
[a_1^{-1}, d_{2k}^{-1}] &= c_{2k}^{p_{2k}} & [a_2^{-1}, c_{2k}^{-1}] &= d_{2k}^{q_{2k}} & [b_1, c_{2k}] = 1 & [b_2, d_{2k}] = 1 & c_k c_{k+1} = 1 \\
\end{align*} \]

\[ \begin{align*}
d_1 d_2 \cdots d_{2k} &= 1 \\
d_2 d_3 \cdots d_{2k-1} &= [c_2, d_{2k}] \\
\ldots & \\
d_{2k+1} d_{2k+2} \cdots d_{3k-1} &= [c_{2k-1}, d_{2k-1}] \cdots [c_{2k-1}, d_{2k-1}] [c_{2k}, d_{2k}] \\
\end{align*} \]
$$\pi_1(X(n, k, \bar{p}, \bar{q}))$$

$$\pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; |$$

\[
\begin{align*}
[b_1^{-1}, d_1^{-1}] = & \quad [c_1^{-1}, b_2] = d_1^{q_1} & [a_1, c_1] = 1 & [a_2, c_1] = 1 \\
a_1 [b_2^{-1}, d_1^{-1}] = & \quad [c_1^{-1}, b_1] = d_2^{q_2} & [a_1, c_2] = 1 & [a_2, c_2] = 1 \\
a_2 [a_1^{-1}, d_1] = b_1 & \quad [d_1^{-1}, b_2^{-1}] = c_1^{p_1} & [a_1, d_1] = 1 & [a_2, d_1] = 1 \\
[a_1^{-1}, d_2] = b_2 & \quad [d_2^{-1}, c_1^{-1}] = c_2^{p_2} & [b_1, c_1] = 1 & [b_2, c_2] = 1 \\
[a_1, b_1][a_2, b_2] = 1, \quad \prod_{j=1}^{2k} [c_j, d_j] = 1 \\
\end{align*}
\]

\[
\begin{align*}
[a_1^{-1}, d_3^{-1}] = & \quad c_3^{p_3} & [a_2^{-1}, c_3^{-1}] = d_3^{q_3} & [b_1, c_3] = 1 & [b_2, d_3] = 1 & c_1 c_{2k} = 1 \\
\cdots & \quad \cdots & \quad \cdots & \quad \cdots & \quad \cdots & \quad \cdots \\
[a_1^{-1}, d_{2k}^{-1}] = & \quad c_{2k}^{p_{2k}} & [a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{q_{2k}} & [b_1, c_{2k}] = 1 & [b_2, d_{2k}] = 1 & c_k c_{k+1} = 1 \\
\end{align*}
\]

\[
\begin{align*}
d_1 d_2 \cdots d_{2k} & = 1 \\
d_2 d_3 \cdots d_{2k-1} & = [c_2, d_{2k}] \\
\cdots & \\
d_k d_{k+1} \cdots d_{2k-1} & = [c_{2k-1}, d_{2k-1}][c_{2k}, d_{2k}] \]
\]
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; | \]

\[
\begin{align*}
[b_1^{-1}, d_1^{-1}] &= \Pi_{j=1}^{2k}[c_j, d_j] = 1 \\
a_1 \left[b_2^{-1}, d_2^{-1}\right] &= a_2 \\
[a_1^{-1}, d_1] &= b_1 \\
[a_2^{-1}, d_2] &= b_2 \\
[a_1, b_1][a_2, b_2] &= 1 \\
[a_1^{-1}, d_3^{-1}] &= c_3^{p_3} \\
\cdots \\
[a_1^{-1}, d_{2k}^{-1}] &= c_{2k}^{p_{2k}} \\
[a_2^{-1}, c_3^{-1}] &= d_3^{p_3} \\
\cdots \\
[a_2^{-1}, c_{2k}^{-1}] &= d_{2k}^{p_{2k}} \\
[b_1, c_3] &= 1 \\
\cdots \\
[b_1, c_{2k}] &= 1 \\
[b_2, d_3] &= 1 \\
\cdots \\
[b_2, d_{2k}] &= 1 \\
c_1 c_{2k} &= 1 \\
c_k c_{k+1} &= 1
\end{align*}
\]
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; | \]

\[ b_1^{-1}, d_1^{-1} = a_1 b_2^{-1}, d_2^{-1} = a_2 \]

\[ a_1^{-1}, d_1 = b_1 \quad a_2^{-1}, d_2 = b_2 \]

\[ [a_1, b_1][a_2, b_2] = 1, \ \prod_{j=1}^{2k} [c_j, d_j] = 1 \]

\[ [a_1^{-1}, d_3^{-1}] = c_3^{p_3} \quad [a_2^{-1}, c_3^{-1}] = d_3^{q_3} \quad [b_1, c_3] = 1 \quad [b_2, d_3] = 1 \quad c_1 c_2 = 1 \]

\[ \cdots \]

\[ [a_1^{-1}, d_{2k}^{-1}] = c_{2k}^{p_{2k}} \quad [a_2^{-1}, c_{2k}^{-1}] = d_{2k}^{q_{2k}} \quad [b_1, c_{2k}] = 1 \quad [b_2, d_{2k}] = 1 \quad c_k c_{k+1} = 1 \]

\[ d_1 d_2 \cdots d_{2k} = 1 \]

\[ d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}] \]

\[ \cdots \]

\[ d_{l+1} d_{l+2} \cdots d_{2k-1} = [c_{2k-l+1}, d_{2k-l+1}] \cdots [c_{2k-1}, d_{2k-1}] [c_{2k}, d_{2k}] \]

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\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; | \]
\[ [b_1^{-1}, d_1^{-1}] = [c_1^{-1}, b_2] = d_1^{q_1} \]
\[ a_1 [b_2^{-1}, d_2^{-1}] = [c_1^{-1}, b_1] = d_2^{q_2} \]
\[ a_2 [a_1^{-1}, d_1] = b_1 \]
\[ [a_2^{-1}, d_2] = [c_2^{-1}, d_1^{-1}] = c_2^{p_2} \]
\[ b_1, c_1 \rangle = \Pi_{j=1}^{2k} [c_j, d_j] = 1 \]
\[ [a_1^{-1}, d_3^{-1}] = c_3^{p_3} \]
\[ [a_2^{-1}, c_3^{-1}] = d_3^{p_3} \]
\[ \cdots \]
\[ [a_1^{-1}, d_{2k}^{-1}] = c_{2k}^{p_2} \]
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\[ b_1, c_{2k} \rangle = \Pi_{j=1}^{2k} [c_j, d_j] = 1 \]
\[ d_1 d_2 \cdots d_{2k} = 1 \]
\[ d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}] \]
\[ \cdots \]
\[ d_{i+1} d_{i+2} \cdots d_{2k-i} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}] [c_{2k}, d_{2k}] \]
\[ \pi_1(X(n, k, \bar{p}, \bar{q})) = \langle c_1, d_1 \cdots c_{2k}, d_{2k}; a_1, b_1, a_2, b_2; \ |
\]

\[ \begin{align*}
[b_1^{-1}, d_1^{-1}] &= [c_1^{-1}, b_2] = d_1^{q_1} & [a_1, c_1] = 1 & [a_2, c_1] = 1 \\
[a_1 [b_2^{-1}, d_2^{-1}] = [c_1^{-1}, b_1] = d_2^{q_2} & [a_1, c_2] = 1 & [a_2, c_2] = 1 \\
[a_2, d_1] = b_1 & [c_1^{-1}, d_1^{-1}] = c_1^{p_1} & [a_1, d_1] = 1 \\
[a_2, d_2] = b_2 & [d_2^{-1}, b_2^{-1}] = c_2^{p_2} & [b_1, c_1] = 1 \\
[a_1, b_1][a_2, b_2] = 1, & \Pi_{j=1}^{2k}[c_j, d_j] = 1
\end{align*} \]

\[ \begin{align*}
[a_1^{-1}, d_3^{-1}] &= c_3^{p_3} & [a_2^{-1}, c_3^{-1}] = d_3^{p_3} & [b_1, c_3] = 1 & [b_2, d_3] = 1 & c_1 c_{2k} = 1 \\
& \cdots & \cdots \\
[a_1^{-1}, d_{2k}^{-1}] &= c_2^{p_2} & [a_2^{-1}, c_{2k}^{-1}] = d_2^{q_2} & [b_1, c_{2k}] = 1 & [b_2, d_{2k}] = 1 & c_k c_{k+1} = 1 \\
1 & \cdots & \cdots \\
& \cdots \\
d_1 d_2 \cdots d_{2k} = 1 & d_2 d_3 \cdots d_{2k-1} = [c_{2k}, d_{2k}] \\
& \cdots \\
d_{i+1} d_{i+2} \cdots d_{2k-i} = [c_{2k-i+1}, d_{2k-i+1}] \cdots [c_{2k-1}, d_{2k-1}][c_{2k}, d_{2k}] \rangle
\]
symplectic 4-manifolds with the various finitely generated fundamental groups

the free group of rank $s \geq 1$ as the fundamental groups

Set $p_3 = \cdots = p_{2k} = 0, p_1 = p_2 = q_1 = \cdots = q_{2k} = 1$ in the above presentation.

Using the identity $c_{2k}^{-1} = c_1$, we rewrite the relation $[a_2^{-1}, c_{2k}^{-1}] = d_{2k}$ as $[a_2^{-1}, c_{2k}^{-1}] = d_{2k}$.

Since $[a_2, c_1] = 1$, we obtain $d_{2k} = 1$. $d_{2k} = 1$ in turn implies $d_1 = 1$ using the relations $d_2d_3\cdots d_{2k-1} = [c_{2k}, d_{2k}]$ and $d_1d_2\cdots d_{2k} = 1$.

Next, using $[b_1^{-1}, d_1^{-1}] = a_1 [a_1^{-1}, d_1] = b_1$ and $[d_1^{-1}, b_2^{-1}] = c_2$, we obtain $a_1 = b_1 = c_1 = 1$.

Since $[c_2^{-1}, b_1] = d_2$ and $[d_2^{-1}, b_1^{-1}] = c_2$, we have $d_2 = c_2 = 1$.

which in turn lead $a_2 = b_2 = 1$.

Next, using the relations $[a_1^{-1}, c_i^{-1}] = d_i$ for any $3 \leq i \leq 2k$, we have $d_3 = d_4 = \cdots = d_{2k} = 1$.

Since $c_{2k-i+1}^{-1} = c_i$ for any $i \leq k$ and $c_1 = c_2 = 1$,

we conclude that $\pi_1(X(n, k, \bar{p}, \bar{q}))$ is a free group of rank $s := k - 2$ generated by $c_3, \cdots, c_k$. 
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Since \([a_2, c_1] = 1\), we obtain \( d_{2k} = 1\). \( d_{2k} = 1\) in turn implies \( d_1 = 1\) using the relations \(d_2d_3\cdots d_{2k-1} = [c_{2k}, d_{2k}]\) and \(d_1d_2\cdots d_{2k} = 1\).

Next, using \([b_1^{-1}, d_1^{-1}] = a_1 [a_1^{-1}, d_1] = b_1\) and \([d_1^{-1}, b_2^{-1}] = c_2\), we obtain \(a_1 = b_1 = c_1 = 1\).

Since \([c_2^{-1}, b_1] = d_2\) and \([d_2^{-1}, b_1^{-1}] = c_2\), we have \(d_2 = c_2 = 1\), which in turn lead \(a_2 = b_2 = 1\).

Next, using the relations \([a_1^{-1}, c_i^{-1}] = d_i\) for any \(3 \leq i \leq 2k\), we have \(d_3 = d_4 = \cdots = d_{2k} = 1\).

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Next, using \([b_1^{-1}, d_1^{-1}] = a_1\) \([a_1^{-1}, d_1] = b_1\) and \([d_1^{-1}, b_2^{-1}] = c_2\), we obtain \( a_1 = b_1 = c_1 = 1\).

Since \([c_2^{-1}, b_1] = d_2\) and \([d_2^{-1}, b_1^{-1}] = c_2\), we have \( d_2 = c_2 = 1\), which in turn lead \( a_2 = b_2 = 1\).

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Since \( c_{2k-1}^{-1} = c_i\) for any \( i \leq k\) and \( c_1 = c_2 = 1\), we conclude that \( \pi_1(X(n, k, p, q))\) is a free group of rank \( s := k - 2 \) generated by \( c_3, \cdots, c_k\).
symplectic 4-manifolds with the various finitely generated fundamental groups

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Since $[c_2^{-1}, b_1] = d_2$ and $[d_2^{-1}, b_1^{-1}] = c_2$, we have $d_2 = c_2 = 1$, which in turn lead $a_2 = b_2 = 1$.

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Since $c_{2k-i+1}^{-1} = c_i$ for any $i \leq k$ and $c_1 = c_2 = 1$, we conclude that $\pi_1(X(n, k, p, q))$ is a free group of rank $s := k - 2$ generated by $c_3, \cdots, c_k$. 

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symplectic 4-manifolds with the various finitely generated fundamental groups

the finite free products of cyclic groups as the fundamental groups

To realize the finite free products of cyclic groups as the fundamental groups, we simply set $p_1 = p_2 = 1, p_3 = \cdots = p_l = 0$, and let $q_i \geq 1$ vary arbitrarily in the above presentation.
symplectic 4-manifolds with the various finitely generated fundamental groups

the finite free products of cyclic groups as the fundamental groups

To realize the finite free products of cyclic groups as the fundamental groups, we simply set $p_1 = p_2 = 1, p_3 = \cdots = p_l = 0$, and let $q_i \geq 1$ vary arbitrarily in the above presentation.
Thank you