Math 1151
Fall 2000 Final Exam Problems

This exam contains 20 multiple-choice questions, worth 7 points each, and 16 written problems, worth various amounts of points, for a total of 300 points.

1. \( \sin(\pi + \theta) = \)
   
   A. \( \sin \theta \)
   B. \(-\sin \theta \)
   C. \( \cos \theta \)
   D. \(-\cos \theta \)
   E. none of the above

2. \( \sin \theta - \sqrt{3} \cos \theta = \)
   
   A. \( 2 \sin(\theta - \pi/3) \)
   B. \( \sqrt{2} \sin(\theta - \pi/3) \)
   C. \( 2 \sin(\theta + \pi/3) \)
   D. \( \sqrt{2} \sin(\theta + \pi/3) \)
   E. none of the above

3. \( \cos^2(\cos(-2\pi/3)) = \)
   
   A. none, or more than one, of the following
   B. \(-2\pi/3\)
   C. \(-\pi/3\)
   D. \(\pi/3\)
   E. \(2\pi/3\)

4. A triangle has sides of length 8, 9, and 12. The angle opposite the side of length 12 is
   
   A. an acute angle (less than 90 degrees)
   B. a right angle (equal to 90 degrees)
   C. an obtuse angle (greater than 90 degrees)
   D. cannot be determined from the given data
   E. none of the above
Note: both questions 5 and 6 refer to the same regular polygon

5. A regular polygon of 14 sides has sides of length 1. Find the diameter of the circumscribed circle. (The circumscribed circle has the same center as the polygon and passes through each of the vertices of the polygon.)
   A. cot (π/14)
   B. csc (π/14)
   C. cot (π/7)
   D. csc (π/7)
   E. none of the above

6. A regular polygon of 14 sides has sides of length 1. Find the diameter of the inscribed circle. (The inscribed circle has the same center as the polygon and is tangent to each of the sides of the polygon.)
   A. cot (π/14)
   B. csc (π/14)
   C. cot (π/7)
   D. csc (π/7)
   E. none of the above

7. In the triangle shown at right, \( \cos \beta \) is equal to
   \[ \begin{align*} 
   A. & \quad \frac{1}{2} \\
   B. & \quad 5/8 \\
   C. & \quad 55/64 \\
   D. & \quad -23/40 \\
   E. & \quad 73/80 
   \end{align*} \]

8. A triangle has sides of lengths 5, 6, and 7. Its area is
   \[ \begin{align*} 
   A. & \quad 6 \cdot \sqrt{3} \\
   B. & \quad 12 \cdot \sqrt{2} \\
   C. & \quad 6 \cdot \sqrt{6} \\
   D. & \quad \sqrt{3} \cdot 5 \cdot 6 \cdot 7 \\
   E. & \quad \text{none of the above} 
   \end{align*} \]

9. \( \sqrt{2} \cdot \sqrt{3} = \)
   \[ \begin{align*} 
   A. & \quad \sqrt{6} \\
   B. & \quad -\sqrt{6} \\
   C. & \quad i \cdot \sqrt{6} \\
   D. & \quad -i \cdot \sqrt{6} \\
   E. & \quad \text{none of the above} 
   \end{align*} \]
10. If $z$ is the complex number $-3-4i$, then the absolute value of $z$ is
   
   A. $3$
   B. $5$
   C. $7$
   D. $3+4i$
   E. none of the above

11. $\tan 2\theta$ is identically equal to
   
   A. none, or more than one, of B through E below
   B. $2 \tan \theta$, but none of C, D, or E
   C. $\frac{2 \tan \theta}{1 + \tan^2 \theta}$, but none of B, D, or E
   D. $\frac{2 \tan \theta}{1 - \tan^2 \theta}$, but none of B, C, or E
   E. $\frac{2 \tan \theta}{\tan^2 \theta - 1}$, but none of B, C, or D

12. $\tan 2\theta$ is identically equal to
   
   A. none, or more than one, of B through E below
   B. $\frac{\sin 4\theta}{1 + \cos 4\theta}$, but none of C, D, or E
   C. $\frac{\sin 4\theta}{1 - \cos 4\theta}$, but none of B, D, or E
   D. $\frac{\cos 4\theta + 1}{\sin 4\theta}$, but none of B, C, or E
   E. $\frac{\cos 4\theta - 1}{\sin 4\theta}$, but none of B, C, or D
13. $S$ is the set of points in the plane whose polar coordinates satisfy the relation

$$r = -3 \sin \theta + 4 \cos \theta.$$ What is $S$?

(Hint: You may wish to write this polar equation in rectangular coordinates. You may also wish to write $-3 \sin \theta + 4 \cos \theta$ in amplitude-phase form.)

A. $S$ is a straight line
B. $S$ is a parabola
C. $S$ is an ellipse, or possibly a circle
D. $S$ is a hyperbola
E. $S$ is none of the above

14. $S$ is the set of points in the plane whose polar coordinates satisfy the relation

$$r = \frac{12}{-3 \sin \theta + 4 \cos \theta}.$$ What is $S$?

(Hint: You may wish to write this polar equation in rectangular coordinates. You may also wish to write $-3 \sin \theta + 4 \cos \theta$ in amplitude-phase form.)

A. $S$ is a straight line
B. $S$ is a parabola
C. $S$ is an ellipse, or possibly a circle
D. $S$ is a hyperbola
E. $S$ is none of the above

15. $S$ is the set of points in the plane whose polar coordinates satisfy the relation

$$r = \frac{3}{2 - 2 \cos \theta}.$$ What is $S$?

(Hint: You may wish to write this polar equation in rectangular coordinates.)

A. $S$ is a straight line
B. $S$ is a parabola
C. $S$ is an ellipse, or possibly a circle
D. $S$ is a hyperbola
E. $S$ is none of the above
Note: the equations in questions 16 - 18 all have the same coefficient matrix

16. The set of simultaneous equations

\[
\begin{align*}
\begin{bmatrix} x & -2z & = 3 \\
y & + z & = 2 \\
x & -y & -3z = 1 \\
\end{bmatrix}
\end{align*}
\]

A. has no solution at all.
B. has exactly one solution.
C. has two or more solutions.
D. a number of solutions which depends upon the value of z.
E. a number of solutions which cannot be determined from these equations.

17. The set of simultaneous equations

\[
\begin{align*}
\begin{bmatrix} x & -2z & = 0 \\
y & + z & = 0 \\
x & -y & -3z = 0 \\
\end{bmatrix}
\end{align*}
\]

A. has no solution at all.
B. has exactly one solution.
C. has two or more solutions.
D. a number of solutions which depends upon the value of z.
E. a number of solutions which cannot be determined from these equations.

18. The set of simultaneous equations

\[
\begin{align*}
\begin{bmatrix} x & -2z & = 2 \\
y & + z & = 1 \\
x & -y & -3z = 0 \\
\end{bmatrix}
\end{align*}
\]

A. has no solution at all.
B. has exactly one solution.
C. has two or more solutions.
D. a number of solutions which depends upon the value of z.
E. a number of solutions which cannot be determined from these equations.
19. Given that $x^2 - 3x + 5$ is a factor of

$$p(x) = x^4 - 3x^3 + 3x^2 + 6x - 10,$$

then the four solutions of the equation $p(x) = 0$

A. are all real numbers
B. are all non-real complex numbers
C. include exactly two real solutions, both of which are positive
D. include exactly two real solutions, one being positive and the other negative
E. include exactly two real solutions, both of which are negative

20. If we write the complex number

$$\frac{-3 - 4i}{-4 + 3i}$$

in the standard form $a + bi$, with $a$ and $b$ being real numbers, then $b$ is equal to

A. -4/3
B. -1
C. 1
D. 4/3
E. none of the above

21. [10 pts.] $\theta$ is an angle for which $180^\circ < \theta < 270^\circ$ and $\tan \theta = 1/3$. Find the exact value of each of the remaining five trigonometric functions of $\theta$.

22. [10 pts.] Find the exact value of $\tan(\alpha + \beta)$, if $\sin \alpha = 4/5$, $0 < \alpha < \pi/2$, $\sin \beta = 5/13$, and $\pi/2 < \beta < \pi$.

Note: the following set of twelve expressions are to be used in answering both parts of problem 23

1. $2 \sin \theta$
2. $2 \sin \theta \cos \theta$
3. $2 \sin^2(\theta/2)$
4. $\sin^2 \theta + \cos^2 \theta$
5. $\sin^3 \theta - \cos^2 \theta$
6. $\cos^2 \theta - \sin^2 \theta$
7. $1 + 2 \cos^2 \theta$
8. $-1 + 2 \cos^2 \theta$
9. $1 - 2 \cos^2 \theta$
10. $1 + 2 \sin^2 \theta$
11. $-1 + 2 \sin^2 \theta$
12. $1 - 2 \sin^2 \theta$

23. [for each part, 4 points, with -1 for each error up to four]
From the set of expressions above, list the numbers of those which are equal to

a) $\sin 2\theta$

b) $\cos 2\theta$
24. The windshield wiper of an automobile is 18 inches long and sweeps through an arc of one-third of a full circle.
   a) [5 pts.] What distance does the tip of the wiper travel in one sweep? (Count only the sweep in one direction, not the return sweep. You may give your answer either in exact form or to the nearest tenth of an inch.)
   b) [5 pts.] It takes 2 seconds for the wiper to travel one sweep. How fast does the tip of the wiper travel?

25. [10 pts.] A triangle has one side of length 32 and adjacent angles of $\pi/6$ and $\pi/4$ radians. Find the length of the side opposite the angle of $\pi/6$.

26. [10 pts.] Establish the trigonometric identity $\frac{\csc \theta}{1 - \cos \theta} = \frac{1 + \cos \theta}{\sin^2 \theta}$

27. [10 pts.] Find, in exact form, all values of $\theta$, in the interval $0 \leq \theta < 2\pi$, for which $\cos 2\theta = \sin \theta$.

28. [9 pts.] Substitute $x = 1 + \cos \theta$ into the expression $\sqrt{2x - x^2}$ and simplify to obtain an expression without radicals.

29. One zero of the polynomial $f(x) = 2x^3 + 3x^2 + 8x - 5$ is a real rational number.
   a) [4 pts.] Find a real zero of $f(x)$.
   b) [3 pts.] Write $f(x)$ as the product of a linear factor and a quadratic factor.
   c) [4 pts.] Find the remaining two zeroes of $f(x)$.
   d) [2 pts.] Write $f(x)$ as a product of linear factors.

30. [10 pts.] Find the solutions of the equation $z^2 = -8 - 8i$. Write them in polar form [that is, as $z = r \cdot (\cos \theta + i \sin \theta)$] and plot the solutions on the complex plane.

31. [10 pts.] Find the equation of the parabola which has its focus at $(0, 3/2)$ and its vertex at $(0, 0)$. Graph this parabola. Include the equation for its directrix and graph that also.
32. [10 pts.] Find the vertices and foci of the ellipse \(4y^2 + 9x^2 = 36\). Draw its graph and carefully indicate the foci and vertices, along with their coordinates.

33. [10 pts.] A Broadway theater has 500 seats. Tickets for the orchestra seats cost $50 each, main floor seats are $35, and balcony seats are $25. If all seats are sold, the gross revenue is $17,100. If all of the main floor and balcony seats, but only half of the orchestra seats are sold, the gross revenue is $14,600. How many seats of each kind are there in this theater?

34. [10 pts.] Find the common difference and the number of terms of an arithmetic sequence, if the first term is -24, the last term is 60, and the sum of the sequence is 900.

35. [10 pts.] The (fictional) campus of the University of Minnesota at Nofreeze-Noexam has a semester system with 15 weeks in each semester, and two semesters per year. The faculty members are paid only for the 30 weeks of classes each year. A faculty member receives a paycheck of $0.01 for the first week, $0.02 for the second week, $0.04 for the third week, and so on, the amount doubling each week. What is the total salary of a faculty member for the 30 weeks of the academic year?

36. You are given the system of simultaneous equations

\[
\begin{align*}
-y &+ 2z = 5 \\
4x &+ 2y - 3z = 2 \\
A &+ z = B
\end{align*}
\]

where \(A\) is either 0 or 1 and \(B\) is also either 0 or 1; \(A\) and \(B\) may be the same or different from each other.

a) [3 pts.] How may \(A\) and \(B\) be chosen so that there is no solution to this system of equations?

b) [3 pts.] How may \(A\) and \(B\) be chosen so that there is exactly one solution to this system of equations? Find that solution (the values of \(x, y,\) and \(z\)).

c) [4 pts.] How may \(A\) and \(B\) be chosen so that there are infinitely many solutions to this system of equations? Given three different possible solutions (that is, three distinct sets of values for \(x, y,\) and \(z\)).