Spectral stability of wave trains in the Kawahara equation

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Abstract

We study the stability of spatially periodic solutions to the Kawahara equation, a fifth order, nonlinear partial differential equation. The equation models the propagation of nonlinear water-waves in the long-wavelength regime, for Weber numbers close to 1/3 where the approximate description through the Korteweg-de Vries (KdV) equation breaks down. Beyond threshold, Weber number larger than 1/3, this equation possess solitary waves just as the KdV approximation. Before threshold, true solitary waves typically do not exist. In this case, the origin is surrounded by a family of periodic solutions and only generalized solitary waves exist which are asymptotic to one of these periodic solutions at infinity. We show that these periodic solutions are spectrally stable at small amplitude.

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1 Introduction

We consider the Kawahara equation

$$\partial_t u = \partial_x^5 u - \epsilon \partial_x^3 u + u \partial_x u, \tag{1.1}$$

in which ϵ is a real parameter. This equation has been derived by Kawahara [11] as a model for water waves in the long-wave regime for moderate values of surface tension, Weber numbers close to 1/3; see [18] for a rigorous result on its validity in the water-wave problem. For such Weber numbers the usual description of long water waves via the Korteweg-de Vries (KdV) equation fails since the cubic term in the linear dispersion relation vanishes and fifth order dispersion becomes relevant at leading order, $\omega(k) = k^5 + \epsilon k^3$. Positive (resp. negative) values of the parameter ϵ in (1.1) correspond to Weber numbers larger (resp. smaller) than 1/3. The linear dispersion relation is monotone for positive ϵ and possesses two extrema for negative ϵ.

The Kawahara equation (1.1) gives an appropriate description of several phenomena observed in the dynamics of the water-wave problem for Weber numbers close to 1/3. We refer, in particular, to questions concerning solitary-wave dynamics. In the water-wave problem, solitary waves exist for Weber numbers larger than 1/3; see [1]. These solitary waves are similar to the ones found in the KdV equation, and therefore often referred to as KdV-like solitary waves or true solitary waves. For Weber numbers smaller than 1/3, true solitary waves typically do not exist [9, 21]. The speed of small-amplitude solitary waves would be close to the group velocity $\omega'(0) = 0$ of large-wavelength perturbations. In this parameter regime, this speed is in resonance with the phase speed of periodic wave trains that correspond to the nontrivial zeroes of the linear dispersion relation ($k = \pm\sqrt{|\epsilon|}$, for $\epsilon < 0$). Therefore, instead of solitary waves that decay to a constant amplitude, we find here generalized solitary waves that decay towards these small periodic waves at their tails [8, 20]. Actually, these generalized solitary waves exist for very small amplitudes of the background periodic wave trains: with respect to the speed of the solitary wave c, the amplitude of the solitary wave scales with c, whereas the amplitude of the background waves can be chosen to be exponentially small in c (but not zero). The same situation, existence and non-existence of solitary waves, and existence of generalized solitary waves, is observed in the Kawahara equation (1.1) for positive and negative values of ϵ, respectively; see [2, 5, 10].

The dynamics of KdV-like solitary waves has been studied in much detail in the context of the KdV equation; see for example [16, 12]. Some stability results are also available for the Kawahara equation [6] and the water-wave problem [4, 13]. We are not aware of results concerning the dynamics of generalized solitary waves beyond their mere existence.

A first step towards a stability analysis for this type of waves is the study of stability properties for the periodic waves arising at their tails. In dissipative systems, stability of periodic waves has been studied both with periodic boundary conditions and with respect to localized perturbations on the real line; see [3, 15, 19] for Evans function approaches to linearized stability and [17, 14] for approaches to a nonlinear stability analysis. In dispersive equations, the Hamiltonian structure can provide energy-type stability criteria when periodic boundary conditions are imposed. Little seems to be known in the physically more interesting case of spatially localized (in particular aperiodic) perturbations of the underlying periodic pattern. In particular, for the Kawahara equation above, we are not aware of any stability result for periodic waves, neither in a spatially periodic setup, nor on the unbounded real line.

The purpose of this paper is to investigate the spectral stability of the periodic wave trains at the tail of the generalized solitary waves of the Kawahara equation (1.1), with respect to localized and bounded perturbations. We prove that periodic travelling waves with speed c are spectrally stable, that is, the spectrum of the linearization about these waves is contained in the imaginary axis, provided their amplitude A satisfies $A = \mathrm{o}(|c|^{5/4})$. Outside this parameter regime, instabilities may occur, but detecting them turns out to be a very difficult task. The spectral analysis of the linearized operator is performed in $L^2(\mathbb{R})$ and $C_{\mathrm{b}}(\mathbb{R})$, implying that the perturbations are either localized or uniformly continuous on the unbounded real line, respectively. In fact the spectra coincide in both cases, and they can be described as the union of the point spectra to a family of operators with periodic boundary conditions using Bloch wave decomposition. We find the point spectra of the latter operators employing mainly perturbation arguments, a careful analysis of the linear dispersion relation, and a bifurcation analysis for small eigenvalues. The method presented here may be applied to other dispersive equations as well. For the KdV equation, our method shows that periodic travelling waves are spectrally stable at small amplitude (see Remark 3.1). The crucial ingredients to the analysis, besides a proper general setup, reduce to a spectral gap argument for large eigenvalues (Lemma 4.2), and the computation of an Evans function type determinant (Lemma 6.7 (vi)).

The paper is organized as follows. The existence of periodic waves is stated in Section 2. In Section 3 we formulate the main result on spectral stability and describe the spectrum of the linearization about the periodic waves with the help of the Bloch wave decomposition. The linear dispersion relation is analyzed in Section 4. We locate the spectrum for Bloch wavenumbers bounded away from zero in Section 5, and for small Bloch wavenumbers in Section 6.

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2 Existence of periodic waves

In this section, we recall the existence result for periodic solutions to the Kawahara equation (1.1).

We consider (1.1) with $\epsilon < 0$, and rescale x, t, and u such that $\epsilon = -1$. We look for solutions travelling with constant speed c from right to left. In a comoving frame, replacing $x + ct$ by x, these solutions are stationary solutions to

$$\partial_t u = -c\partial_x u + \partial_x^5 u + \partial_x^3 u + u\partial_x u, \tag{2.1}$$

and therefore satisfy

$$-cu + \partial_x^4 u + \partial_x^2 u + \frac{1}{2}u^2 = C,$$

in which C is an arbitrary constant. As a remnant of the Galilean invariance of the equation (2.1), $u \mapsto u + \alpha$, $c \mapsto c + \alpha$, in the equation above the constant C can be set to 0. In the following, we therefore restrict to

$$-cu + \partial_x^4 u + \partial_x^2 u + \frac{1}{2}u^2 = 0. \tag{2.2}$$

We point out that this choice of coordinates and scalings is actually used in the construction of generalized solitary waves. They are found for $c > 0$, as homoclinic orbits connecting periodic orbits of small amplitude. In the next theorem we state the existence of these periodic orbits, together with some properties which are needed in our stability analysis. We give a short proof of this result for (2.2), and refer to [10, Chapter 4] for an existence result in a more general context.

Theorem 1 *There exist positive constants c_0 and a_0 such that for any $c \in (-c_0, c_0)$, the equation (2.2) has a one-parameter family of even, periodic solutions $(\varphi_{a,c})_{a\in(-a_0,a_0)}$ of the form*

$$\varphi_{a,c}(x) = p_{a,c}(kx), \tag{2.3}$$

in which $k = k(a, c)$ and $p_{a,c}$ is 2π-periodic in its argument. Moreover, the following properties hold.

(i) *The map $k : (-a_0, a_0) \times (-c_0, c_0) \to \mathbb{R}$ is analytic and*

$$k(a, c) = k_0(c) + c\widetilde{k}(a, c),$$

where

$$k_0(c) = \left(\frac{1 + \sqrt{1 + 4c}}{2}\right)^{1/2}, \quad \widetilde{k}(a, c) = \sum_{n\geq 1} \widetilde{k}_{2n}(c)\, a^{2n}, \quad \left|\widetilde{k}_{2n}(c)\right| \leq \frac{K_0}{\rho_0^{2n}}, \tag{2.4}$$

for any a, c, and some positive constants K_0 and $\rho_0 > a_0$.

(ii) *The map* $(a,c)\mapsto p_{a,c}:(-a_0,a_0)\times(-c_0,c_0)\to H^4(\mathbb{R}/2\pi\mathbb{Z},\mathbb{R})$ *is analytic and*

$$p_{a,c}(z)=ac\cos(z)+c\sum_{\substack{n,\,m\geq 0,\,n+m\geq 2\\ n-m\neq\pm 1}}\widetilde{p}_{n,m}(c)\,\mathrm{e}^{\mathrm{i}(n-m)z}a^{n+m},\tag{2.5}$$

in which $\widetilde{p}_{n,m}(c)$ *are real numbers such that* $\widetilde{p}_{n,m}(c)=\widetilde{p}_{m,n}(c)$ *with*

$$\left|\widetilde{p}_{n,m}(c)\right|\leq\frac{C_0}{\rho_0^{n+m}},\tag{2.6}$$

for any c*, and some positive constant* C_0*.*

(iii) *The Fourier coefficients* $\widehat{p}_q(a,c)$ *of the* 2π*-periodic function* $p_{a,c}$*,*

$$p_{a,c}(z)=\sum_{q\in\mathbb{Z}}\widehat{p}_q(a,c)\mathrm{e}^{\mathrm{i}qz},$$

are real and satisfy $\widehat{p}_0(a,c)=\mathrm{O}(ca^2)$ *and* $\widehat{p}_q(a,c)=\mathrm{O}(c|a|^{|q|})$*,* $q\neq 0$*, as* $a\to 0$*. Moreover, the map* $a\mapsto\widehat{p}_q(a,c)$ *is even (resp. odd) for even (resp. odd) values of* q*.*

Proof. We look for periodic solutions to (2.2) with wavenumber k. We normalize the period to 2π by setting $z=kx$, and find the equation

$$-cu+k^4\partial_z^4u+k^2\partial_z^2u+\frac{1}{2}u^2=0.\tag{2.7}$$

The linear part of this equation defines a closed linear operator $L(c,k)$ on the space $L^2(\mathbb{R}/2\pi\mathbb{Z})$ of 2π–periodic functions which are locally square integrable, with domain $H^4(\mathbb{R}/2\pi\mathbb{Z})$. For $c\sim 0$, $c\neq 0$ this operator has a two-dimensional kernel spanned by $e^{\pm\mathrm{i}z}$ if $k=k_0(c)$, where $k_0(c)$ is given by (2.4). We construct periodic solutions to (2.7) by using a Lyapunov-Schmidt reduction for k close to $k_0(c)$.

We set $k=k_0(c)+c\widetilde{k}$, and

$$u(z)=\frac{1}{2}Ace^{\mathrm{i}z}+\frac{1}{2}\bar{A}ce^{-\mathrm{i}z}+cv(z),\tag{2.8}$$

in which $A\in\mathbb{C}$, and $v\in H^4(\mathbb{R}/2\pi\mathbb{Z})$ satisfies

$$\widehat{v}_{\pm 1}=\frac{1}{2\pi}\int_{-\pi}^{\pi}v(z)e^{\mp\mathrm{i}z}\mathrm{d}z=0.$$

By substituting (2.8) into (2.7) we obtain an equation of the form

$$L_cv=N(v,A,\bar{A},\widetilde{k},c),\tag{2.9}$$

where

$$L_cv=\frac{k_0(c)^4\partial_z^4v+k_0(c)^2\partial_z^2v-cv}{c},$$

and $N(0,0,0,\widetilde{k},c)=0$. Here the operators L_c are defined for $c\neq 0$ only. We transform (2.9) into a system which is well defined at $c=0$ as well.

The kernel of L_c is two-dimensional and spanned by $e^{\pm \mathrm{i}z}$. We denote by P the spectral projection onto the kernel of L_c, defined for any $u\in L^2(\mathbb{R}/2\pi\mathbb{Z})$ by $Pu=\widehat{u}_1 e^{\mathrm{i}z}+\widehat{u}_{-1}e^{-\mathrm{i}z}$, where $\widehat{u}_{\pm 1}$ are the Fourier coefficients of u corresponding to the modes $e^{\pm \mathrm{i}z}$. Since $Pv=0$, the equation (2.9) is equivalent to

$$L_c v = (\mathrm{id}-P)N(v,A,\bar{A},\widetilde{k},c), \quad 0 = PN(v,A,\bar{A},\widetilde{k},c). \tag{2.10}$$

The restriction of L_c to $(\mathrm{id}-P)H^4(\mathbb{R}/2\pi\mathbb{Z})$, the space orthogonal to the kernel, has a bounded inverse given by

$$L_c^{-1}v = -\widehat{v}_0 + \sum_{|n|\geq 2} \frac{c\widehat{v}_n}{k_0(c)^4 n^4 - k_0(c)^2 n^2 - c} e^{\mathrm{i}nz},$$

for any $v\in(\mathrm{id}-P)L^2(\mathbb{R}/2\pi\mathbb{Z})$, $v=\sum_{n\neq\pm 1}\widehat{v}_n e^{\mathrm{i}nz}$. This formula shows that the inverse L_c^{-1} can be extended to $c=0$, and that the operators L_c^{-1} form a family of bounded operators depending analytically upon c. Therefore, the system (2.10) is equivalent to

$$v = L_c^{-1}(\mathrm{id}-P)N(v,A,\bar{A},\widetilde{k},c) \tag{2.11}$$

$$0 = PN(v,A,\bar{A},\widetilde{k},c) \tag{2.12}$$

which is well defined for any c in some neighborhood of the origin.

With the help of the implicit function theorem, we may solve (2.11). We find a unique solution $v=V_*(A,\bar{A},\widetilde{k},c)\in(\mathrm{id}-P)H^4(\mathbb{R}/2\pi\mathbb{Z})$ that depends analytically upon $(A,\bar{A},\widetilde{k},c)$ in a neighborhood of the origin of $\mathrm{diag}(\mathbb{C}^2)\times\mathbb{R}^2$. Here, $\mathrm{diag}(\mathbb{C}^2)=\{(z,\bar{z})\in\mathbb{C}^2\}\sim\mathbb{R}^2$. Furthermore, the uniqueness of this solution implies that

$$V_*(0,0,\widetilde{k},c)=0.$$

The invariance of (2.7) under translations $z\mapsto z+\zeta$ and under the reflection $z\mapsto -z$ enforces the relations

$$V_*(A,\bar{A},\widetilde{k},c)(z+\alpha) = V_*(Ae^{\mathrm{i}\alpha},\bar{A}e^{-\mathrm{i}\alpha},\widetilde{k},c)(z), \quad V_*(A,\bar{A},\widetilde{k},c)(-z) = V_*(\bar{A},A,\widetilde{k},c)(z).$$

By substituting the solution $V_*(A,\bar{A},\widetilde{k},c)$ of (2.11) into (2.12) and using the explicit formula for the projection P we obtain the solvability conditions

$$J_c(A,\bar{A},\widetilde{k},c) = \int_{-\pi}^{\pi} \frac{Ae^{\mathrm{i}s}+\bar{A}e^{-\mathrm{i}s}}{2} N(V_*,A,\bar{A},\widetilde{k},c)(s)\mathrm{d}s = 0 \tag{2.13}$$

$$J_s(A,\bar{A},\widetilde{k},c) = \int_{-\pi}^{\pi} \frac{Ae^{\mathrm{i}s}-\bar{A}e^{-\mathrm{i}s}}{2\mathrm{i}} N(V_*,A,\bar{A},\widetilde{k},c)(s)\mathrm{d}s = 0. \tag{2.14}$$

The invariances of (2.7) mentioned above imply that J_c and J_s satisfy

$$J_c(Ae^{\mathrm{i}\alpha},\bar{A}e^{-\mathrm{i}\alpha},\widetilde{k},c)=J_c(A,\bar{A},\widetilde{k},c)=J_c(\bar{A},A,\widetilde{k},c),$$
$$J_s(Ae^{\mathrm{i}\alpha},\bar{A}e^{-\mathrm{i}\alpha},\widetilde{k},c)=J_s(A,\bar{A},\widetilde{k},c)=-J_s(\bar{A},A,\widetilde{k},c).$$

By taking $\alpha=-2\arg A$ in the equalities for J_s we obtain

$$J_s(\bar{A},A,\widetilde{k},c)=J_s(A,\bar{A},\widetilde{k},c)=-J_s(\bar{A},A,\widetilde{k},c),$$

so that $J_s\equiv 0$ and the condition (2.14) is always satisfied.

In the equalities for J_c we set $\alpha=-\arg A$ and find $J_c(A,\bar{A},\widetilde{k},c)=J_c(|A|,|A|,\widetilde{k},c)$, so that the solvability condition (2.13) becomes

$$J_c(a,a,\widetilde{k},c)=0,$$

where a belongs to a neighborhood of the origin in $\mathbb{R}$. By using the explicit formula for N and the symmetries of V_* we find that

$$J_c(a,a,\widetilde{k},c)=-a^2\left(\pi(4k_0(c)^3-2k_0(c))\widetilde{k}+J(a,\widetilde{k},c)\right),$$

in which J is analytic in its arguments, even with respect to a, and satisfies $J(0,0,c)=\partial_{\widetilde{k}}J(0,0,c)=0$. By arguing again with the implicit function theorem, we obtain a solution $\widetilde{k}_*(a,c)$ of $J_c(a,a,\widetilde{k},c)=0$ for small a and c. Furthermore, $\widetilde{k}_*(a,c)$ is even in a, so that the function $k(a,c)=k_0(c)+c\widetilde{k}_*(a,c)$ verifies the properties stated in part (i) of the theorem.

The arguments above show that (2.11)–(2.12) has a unique solution

$$(v,\widetilde{k})=(V_*(A,\bar{A},\widetilde{k}_*(|A|,c),c),\widetilde{k}_*(|A|,c)),$$

for any small complex number A and real c. Substitution of $v=V_*(A,\bar{A},\widetilde{k}_*(|A|,c),c)$ into (2.8) yields a 2π-periodic solution to (2.7). The periodic solutions $p_{a,c}$ in the theorem are the ones obtained by restricting to $A\in\mathbb{R}$,

$$p_{a,c}(z)=ac\cos(z)+cv_{a,c}(z),\quad v_{a,c}(z)=V_*(a,a,\widetilde{k}_*(a,c),c).$$

The properties of $p_{a,c}$ stated in (ii) are deduced from the analyticity and the symmetries of the function $V_*(A,\bar{A},\widetilde{k},c)$. Finally, (iii) is a consequence of the expansion (2.5). ■

Remark 2.1 (i) *Note that in the result and proof, we restricted the amplitude $A\leq ca_0$ to a small sector of (A,c)–plane. This is sufficient for our purpose of a description of the waves at the tail of the generalized solitary waves. However, it is not difficult to derive a full bifurcation diagram for the existence of periodic solutions. The unfolding near $c=0$*

can be studied using Lyapunov-Schmidt reduction on the kernel of the linearization in the subspace of even functions, spanned by $\cos x$ *and* 1. *The bifurcation equations for* $u = a_0 + a_1 \cos x + h.o.t$ *can be solved for* a_1*:*

$$a_1^2 = -2(\delta k^2 - c^2) + \mathrm{O}(3), \quad a_0 = c - 2\delta k + a_1^2/96 + \mathrm{O}(2),$$

where $\delta k = k - 1$. *In particular, there are unique periodic solutions inside two sectors in the* $(c, \delta k)$*–plane bounded by two analytic curves with tangent vectors* $c = \pm\delta k$ *in* $c = \delta k = 0$. *However, we do not know whether all these periodic solutions are stable. Our stability results below are concerned with waves in a sufficiently small sector along the boundary of the two cones.*

(ii) *The formula (2.5) shows, in particular, that* $p_{-a,c}(z) = p_{a,c}(z + \pi)$, *so the periodic solutions found for* $a < 0$ *are translations by a half-period of the ones for* $a > 0$. *Therefore, we can, without loss of generality, take* $a \geq 0$.

(iii) *In Theorem 1, the family of periodic waves is parameterized by the amplitude* a *(see (2.5)). We may equivalently parameterize the family by the wavenumber* k, *since the map* $\Psi : (0, a_0) \times (-c_0, c_0) \to \mathbb{R}^2$ *defined through*

$$\Psi(a, c) = (k(a, c), c),$$

is locally invertible and injective on $(0, a_0) \times (0, c_0)$ *and on* $(0, a_0) \times (-c_0, 0)$.

3 The main result

We state our main result, that claims spectral stability of the family of periodic solutions which are described in Theorem 1, and gives a detailed characterization of the spectrum of the linearization at these periodic solutions.

We are interested in the stability of the periodic solutions in Theorem 1 as steady solutions to the evolution problem (2.1). We set

$$u(x, t) = \varphi_{a,c}(x) + v(x, t),$$

and obtain the evolution problem for the perturbations v,

$$\partial_t v = \mathcal{A}_{a,c} v + \mathcal{N}(v),$$

in which the linear terms $\mathcal{A}_{a,c} v$ and the nonlinear terms $\mathcal{N}(v)$ are given by

$$\mathcal{A}_{a,c} v = \partial_x^5 v + \partial_x^3 v - c\partial_x v + \partial_x(\varphi_{a,c} v), \quad \mathcal{N}(v) = v\partial_x v.$$

We regard $\mathcal{A}_{a,c}$ as a linear operator acting on a Banach space $\mathcal{X}$ of x-dependent functions. There are several possible choices for this Banach space, each of them corresponds to a certain class of perturbations v. Here, we take either $\mathcal{X} = L^2(\mathbb{R})$, for spatially localized perturbations, or $\mathcal{X} = C_{\mathrm{b}}(\mathbb{R})$, for bounded uniformly continuous perturbations.

The main result in this paper is the following theorem on the spectrum of the linear operator $\mathcal{A}_{a,c}$. Spectral stability of the periodic solutions $\varphi_{a,c}$ for sufficiently small a and c is an immediate consequence.

Theorem 2 *There exist positive constants c_1 and a_1, such that for any $c \in (-c_1, c_1)$ and $a \in (-a_1|c|^{1/4}, a_1|c|^{1/4})$ the spectrum of the linear operator $\mathcal{A}_{a,c}$ in $L^2(\mathbb{R})$, or $C_{\mathrm{b}}(\mathbb{R})$, lies on the imaginary axis.*

Remark 3.1 (Wave trains in the KdV equation) *A similar result of spectral stability can be obtained for the periodic travelling waves of the KdV equation*

$$\partial_t u = \partial_x^3 u + u\partial_x u.$$

In this case the analysis is much simpler, in particular since the speed of the wave can be scaled out of the equation. Indeed, in the frame of reference moving with the speed c of the wave the equation becomes

$$\partial_t u = -c\partial_x u + \partial_x^3 u + u\partial_x u,$$

in which $x + ct$ has been replaced by x just like in the equation (2.1). In contrast to (2.1), here we can eliminate c by setting $u(x,t) = |c|v(|c|^{1/2}x, |c|^{3/2}t)$. We find the rescaled equation

$$\partial_t v = \partial_x^3 v - \mathrm{sign}(c)\partial_x v + v\partial_x v, \tag{3.1}$$

in which we may take $\mathrm{sign}(c) = -1$, since the sign of the coefficient of $\partial_x v$ in (3.1) is changed by the translation $v \mapsto v + 2\mathrm{sign}(c)$. The steady solutions of (3.1) are easily found – and well-known – by a phase-plane analysis for the corresponding ODE,

$$\partial_x^2 v + v + \frac{1}{2}v^2 = 0.$$

One finds a homoclinic connection to -2,

$$h(x) = -2 + 3\,\mathrm{sech}^2\left(\frac{x}{2}\right),$$

which corresponds to the KdV solitary wave, and a one-parameter family of periodic orbits surrounding the origin and filling the region inside this homoclinic orbit,

$$\varphi_a(x) = p_a(k(a)x), \qquad p_a(z) = a\cos z + \mathrm{O}(a^2), \qquad k(a) = 1 + \mathrm{O}(a^2),$$

in which p_a is 2π-periodic in its argument. (Notice that the choice $\mathrm{sign}(c) = 1$ *is preferred when solitary waves are analyzed, since then the corresponding homoclinic orbit is a connection to the origin.) Spectral stability of the wave trains in the KdV equation can be studied along the lines of this paper, giving stability for* all *small amplitude waves. This confirms formal results by Whitham for large-wavelength perturbations, derived by means of an asymptotic expansion of the associated Lagrangian (the so-called variational method). We emphasize that the proof for the KdV equation is much simpler, partly because of the simpler, cubic dispersion relation, but also because a scaling symmetry allows for a parameterization of the waves by only one parameter, the amplitude a, largely simplifying the Blochwave analysis in Section 6.*

The remainder of this paper is occupied with the proof of Theorem 2. The proof relies on a bifurcation analysis for the linearized problem. Since the coefficients of the operator $\mathcal{A}_{a,c}$ are periodic, the spectrum can be described as the union of point spectra to a family of operators with compact resolvent using Bloch wave decomposition. We then trace the point spectrum out of the bifurcation point $a = c = 0$ for all possible Bloch wavenumbers γ.

We first normalize the period of the waves to 2π by replacing kx by x. In the scaled coordinates, we have to study the operator

$$\mathcal{A}_{a,c}v = k^5\partial_x^5 v + k^3\partial_x^3 v - ck\partial_x v + k\partial_x(p_{a,c}v), \tag{3.2}$$

in which $k = k(a, c)$ and $p_{a,c}$ are given by Theorem 1. In particular, both k and $p_{a,c}$ depend analytically upon a and c. We consider the operator $\mathcal{A}_{a,c}$ as a closed linear operator in $L^2(\mathbb{R})$ with domain of definition $H^5(\mathbb{R})$, and in $C_\mathrm{b}(\mathbb{R})$ with domain $C^5_\mathrm{b}(\mathbb{R})$. We denote by $\mathrm{spec}_{L^2}(\mathcal{A}_{a,c})$ and $\mathrm{spec}_{C_\mathrm{b}}(\mathcal{A}_{a,c})$ its spectrum in $L^2(\mathbb{R})$ and $C_\mathrm{b}(\mathbb{R})$, respectively. The following result, which holds for general differential operators with periodic coefficients, shows that the two spectra coincide. Moreover, it gives a simple characterization of these sets in terms of the kernel of $\mathcal{A}_{a,c} - \lambda\mathrm{id}$ in $C_\mathrm{b}(\mathbb{R})$; see for example [3] for a proof of this result in the case of $C_b(\mathbb{R})$, which can be easily adapted to $L^2(\mathbb{R})$.

Proposition 3.2 *The following equalities hold*

$$\mathrm{spec}_{L^2}(\mathcal{A}_{a,c}) = \mathrm{spec}_{C_\mathrm{b}}(\mathcal{A}_{a,c}) = \{\lambda \in \mathbb{C}\,;\ \ker_{C_\mathrm{b}}(\mathcal{A}_{a,c} - \lambda\mathrm{id}) \neq \{0\}\}.$$

Furthermore, $\lambda \in \mathbb{C}$ belongs to one of these sets if and only if there exists a nonzero function $v \in C^5_\mathrm{b}(\mathbb{R})$ of the form $v(x) = u(x)\mathrm{e}^{\mathrm{i}\gamma x}$, with u 2π–periodic and $\gamma \in [-\frac{1}{2}, \frac{1}{2})$, such that $\mathcal{A}_{a,c}v - \lambda v = 0$.

This proposition shows that in order to determine the spectrum of $\mathcal{A}_{a,c}$, we have to find all values of $\lambda \in \mathbb{C}$ such that there exists a nontrivial solution to the linear non-autonomous ordinary differential equation

$$\lambda v=\mathcal{A}_{a,c}v=k^5\partial_x^5v+k^3\partial_x^3v-ck\partial_xv+k\partial_x(p_{a,c}v), \tag{3.3}$$

of the form $v(x)=u(x)\mathrm{e}^{\mathrm{i}\gamma x}$, with a 2π–periodic function u and some real parameter $\gamma\in[-\frac{1}{2},\frac{1}{2})$. Equivalently, we may determine all values $\lambda\in\mathbb{C}$ such that there is a nontrivial 2π-periodic solution to the equation

$$\lambda u=k^5(\partial_x+\mathrm{i}\gamma)^5u+k^3(\partial_x+\mathrm{i}\gamma)^3u-ck(\partial_x+\mathrm{i}\gamma)u+k(\partial_x+\mathrm{i}\gamma)(p_{a,c}u), \tag{3.4}$$

for some $\gamma\in[-\frac{1}{2},\frac{1}{2})$. We denote by $L_{\gamma,a,c}$ the linear operator defined by the right hand side of the equation (3.4). We consider this operator as a closed operator in the space $L^2(\mathbb{R}/2\pi\mathbb{Z})$ with domain $H^5(\mathbb{R}/2\pi\mathbb{Z})$. The following result is then a direct consequence of Proposition 3.2.

Corollary 3.3 *Consider the linear operator $L_{\gamma,a,c}:H^5(\mathbb{R}/2\pi\mathbb{Z})\to L^2(\mathbb{R}/2\pi\mathbb{Z})$ defined by*

$$L_{\gamma,a,c}u=k^5(\partial_x+\mathrm{i}\gamma)^5u+k^3(\partial_x+\mathrm{i}\gamma)^3u-ck(\partial_x+\mathrm{i}\gamma)u+k(\partial_x+\mathrm{i}\gamma)(p_{a,c}u),$$

for $\gamma\in[-\frac{1}{2},\frac{1}{2})$ and $\lambda\in\mathbb{C}$. Then the spectrum of $L_{\gamma,a,c}$ satisfies

$$\mathrm{spec}(L_{\gamma,a,c})=\{\lambda\in\mathbb{C}\,;\ \ker(L_{\gamma,a,c}-\lambda\mathrm{id})\neq\{0\}\}.$$

Moreover, we have that

$$\mathrm{spec}_{L^2}(\mathcal{A}_{a,c})=\mathrm{spec}_{C_\mathrm{b}}(\mathcal{A}_{a,c})=\bigcup_{\gamma\in[-\frac{1}{2},\frac{1}{2})}\mathrm{spec}(L_{\gamma,a,c}).$$

From now on we denote by $\mathrm{spec}(\mathcal{A}_{a,c})$ the spectrum of $\mathcal{A}_{a,c}$ in both $L^2(\mathbb{R})$ and C_b, since the two spectra coincide. This corollary shows that in order to determine the spectrum of the operators $\mathcal{A}_{a,c}$ we have to solve (3.4) for $\lambda\in\mathbb{C}$ and $\gamma\in[-\frac{1}{2},\frac{1}{2})$ in $H^5(\mathbb{R}/2\pi\mathbb{Z})$. Theorem 2 then follows from the following proposition.

Proposition 3.4 (i) *For every $\gamma^*\in(0,\frac{1}{2})$ there exists $c_2^*>0$* [1] *such that for any $c\in\big(-c_2^*,c_2^*\big)$, $a\in(-a_0,a_0)$, and $\gamma\in[-\frac{1}{2},\frac{1}{2})$ with $|\gamma|\geq\gamma^*$, we have*

$$\mathrm{spec}(L_{\gamma,a,c})\subset\mathrm{i}\mathbb{R}.$$

(ii) *There exist positive constants c_3,a_3,γ_3 such that for any $c\in(-c_3,c_3)$, $\gamma\in(-\gamma_3,\gamma_3)$, and $a\in(-a_3|c|^{1/4},a_3|c|^{1/4})$, we have*

$$\mathrm{spec}(L_{\gamma,a,c})\subset\mathrm{i}\mathbb{R}.$$

[1] *For the rest of the paper we distinguish constants depending upon $\gamma^*\in(0,\frac{1}{2})$ by the superscript *.*

To prove this proposition we regard the operator $L_{\gamma,a,c}$ as a perturbation of a skew adjoint operator with constant coefficients $L^0_{\gamma,a,c}$ by an operator $L^1_{\gamma,a,c}$ with small periodic coefficients, more precisely, we write

$$L_{\gamma,a,c} = L^0_{\gamma,a,c} + L^1_{\gamma,a,c},$$

with

$$\begin{aligned} L^0_{\gamma,a,c}u &= k^5(\partial_x + \mathrm{i}\gamma)^5 u + k^3(\partial_x + \mathrm{i}\gamma)^3 u - ck(\partial_x + \mathrm{i}\gamma)u, \\ L^1_{\gamma,a,c}u &= k(\partial_x + \mathrm{i}\gamma)(p_{a,c}u). \end{aligned}$$

Note that L^0 and L^1 depend implicitly on a and c via $k = k(a,c)$. Section 4 is devoted to the study of $L^0_{\gamma,a,c}$. In Section 5 we compute the spectrum of $L_{\gamma,a,c}$ for γ bounded away from zero (Proposition 3.4 (i)), and in Section 6 we consider values of γ close to zero (Proposition 3.4 (ii)).

Remark 3.5 (i) *The spectrum of $L_{\gamma,a,c}$ is symmetric with respect to the imaginary axis. More precisely, we have*

$$\lambda \in \mathrm{spec}(L_{\gamma,a,c}) \quad \Leftrightarrow \quad -\overline{\lambda} \in \mathrm{spec}(L_{\gamma,a,c}) \quad \Leftrightarrow \quad -\lambda \in \mathrm{spec}(L_{-\gamma,a,c}),$$

where we exploited that $p_{a,c}$ is real and even in the first and second relation, respectively.

(ii) *Assume that $a \neq 0$ and $c \neq 0$. Then, the derivative with respect to x of $p_{a,c}$ satisfies (3.4) with $\gamma = 0$ and $\lambda = 0$, due to translation invariance of (2.1) in x. Consequently, 0 always belongs to the spectrum of $L_{0,a,c}$ and $\mathcal{A}_{a,c}$. Furthermore, the geometric multiplicity of 0 as an eigenvalue of $L_{0,a,c}$ is at least two, and its algebraic multiplicity at least three. A principal eigenvector is given by the constant function $1/k(a,c)$, $k(a,c)$ being the wavenumber of the periodic wave. A second principal eigenvector is the derivative of the periodic wave with respect to the speed c in the parameterization of the family of periodic solutions by the wavenumber k and the speed c (see Remark 2.1). We write $p_{a,c} = q_{k,c}$, so that $q_{k,c}$ satisfies*

$$k^5\partial_x^5 q_{k,c} + k^3\partial_x^3 q_{k,c} - ck\partial_x q_{k,c} + kq_{k,c}\partial_x q_{k,c} = 0.$$

By differentiating this equality with respect to c we find

$$L_{0,k,c}\,(\partial_c q_{k,c}) = k\partial_x q_{k,c},$$

so $\partial_c q_{k,c}/k$ is a principal vector to the eigenvalue 0 of $L_{0,a,c}$. Of course, the difference $\partial_c q_{k,c} - 1$ belongs to the kernel of $L_{0,k,c}$, which is then two-dimensional, at least, since $\partial_c q_{k,c} - 1$ is an even function and the derivative with respect to x of $p_{a,c}$ an odd function. We prove later, Lemma 6.7, that for $a \neq 0$ and $c \neq 0$, the geometric and algebraic multiplicities of 0 are precisely two and three, respectively.

4 Study of the unperturbed operator

This section is devoted to the study of the unperturbed operator $L^0_{\gamma,a,c}$. Classical results from linear operator theory and a standard Fourier analysis show that $L^0_{\gamma,a,c}$ has the following properties.

Lemma 4.1 *Assume that $c \in (-c_0, c_0)$, $a \in (-a_0, a_0)$, and $\gamma \in [-\frac{1}{2}, \frac{1}{2})$.*

(i) *The linear operator $L^0_{\gamma,a,c}$ is a closed linear operator in $L^2(\mathbb{R}/2\pi\mathbb{Z})$ with dense domain $H^5(\mathbb{R}/2\pi\mathbb{Z})$.*

(ii) *$L^0_{\gamma,a,c}$ is skew-adjoint and has compact resolvent.*

(iii) *The spectrum of $L^0_{\gamma,a,c}$ consists of a countable number of purely imaginary eigenvalues with finite multiplicities.*

(iv) *Any eigenvalue $\lambda \in \operatorname{spec}(L^0_{\gamma,a,c})$ is semi-simple, and it is given by $\lambda = \mathrm{i}\omega^0_n(\gamma, a, c)$ in which*

$$\omega^0_n(\gamma, a, c) = k^5(\gamma + n)^5 - k^3(\gamma + n)^3 - ck(\gamma + n), \tag{4.1}$$

for some $n \in \mathbb{Z}$. The unique (up to scalar multiples) eigenvector associated to $\mathrm{i}\omega^0_n(\gamma, a, c)$ is $\mathrm{e}^{\mathrm{i}nx}$.

(v) *The resolvent operator $R^0_{\gamma,a,c}(\lambda) = (\lambda \mathrm{id} - L^0_{\gamma,a,c})^{-1}$ is given by*

$$R^0_{\gamma,a,c}(\lambda)(f) = \sum_{n\in\mathbb{Z}} \frac{\widehat{f}_n \mathrm{e}^{\mathrm{i}nx}}{\lambda - \mathrm{i}\omega^0_n(\gamma, a, c)},$$

where $f \in L^2(\mathbb{R}/2\pi\mathbb{Z})$, and $\widehat{f}_n$ represent the Fourier coefficients of f, $f = \sum_{n\in\mathbb{Z}} \widehat{f}_n \mathrm{e}^{\mathrm{i}nx}$.

A more detailed analysis of the dispersion relation (4.1) yields the following result on the relative position on the imaginary axis of the eigenvalues $\lambda = \mathrm{i}\omega^0_n(\gamma)$ in the spectrum of $L^0_{\gamma,a,c}$.

Lemma 4.2 (i) *There exist positive constants c_4 and m_4 such that for any $n, p \in \mathbb{Z}$ with $n \neq p$ and $|n| \geq 2$ or $|p| \geq 2$, and for any $c \in (-c_4, c_4)$, $a \in (-a_0, a_0)$, and $\gamma \in [-\frac{1}{2}, \frac{1}{2})$, the following inequality holds:*

$$|\omega^0_n(\gamma, a, c) - \omega^0_p(\gamma, a, c)| \geq m_4(1 + |p|)(1 + |n|).$$

(ii) *For every $\gamma^* \in (0, \frac{1}{2})$, there exist $c^*_5 > 0$ and $m^*_5 > 0$ such for any $n, p \in \mathbb{Z}$ with $n \neq p$, any $c \in (-c^*_5, c^*_5)$, $a \in (-a_0, a_0)$, and $\gamma \in [-\frac{1}{2}, \frac{1}{2})$ with $\gamma^* \leq |\gamma|$, the following inequality holds:*

$$|\omega^0_n(\gamma, a, c) - \omega^0_p(\gamma, a, c)| \geq m^*_5(1 + |p|)(1 + |n|).$$

Using Fourier analysis, we deduce immediately from this lemma the following corollary.

Corollary 4.3 *Let λ^0 be an eigenvalue of $L^0_{\gamma,a,c}$, $\lambda^0 = \mathrm{i}\omega^0_n(\gamma,a,c)$, $n\in\mathbb{Z}$, and assume that*

$$\text{either} \qquad |n|\geq 2, \quad \gamma\in[-\tfrac{1}{2},\tfrac{1}{2}), \quad a\in(-a_0,a_0), \quad c\in(-c_4,c_4), \tag{H1}$$

$$\text{or} \qquad \gamma\in[-\tfrac{1}{2},\tfrac{1}{2}), \quad |\gamma|\geq\gamma^*>0, \quad a\in(-a_0,a_0), \quad c\in(-c_5^*,c_5^*). \tag{H2}$$

Then, the following properties hold.

(i) *λ^0 is a simple eigenvalue and the kernel of $L^0_{\gamma,a,c}-\lambda^0\mathrm{id}$ is spanned by $\mathrm{e}^{\mathrm{i}nx}$. The corresponding spectral projection Π_n is given by*

$$\Pi_n(f) = \frac{1}{2\pi}\left(\int_{-\pi}^{\pi} f(s)\mathrm{e}^{-\mathrm{i}ns}\,\mathrm{d}s\right)\mathrm{e}^{\mathrm{i}nx}.$$

(ii) *The restriction of $\mathrm{i}\omega^0_n(\gamma,a,c)\mathrm{id}-L^0_{\gamma,a,c}$ to the space orthogonal to its kernel, $H_n = (\mathrm{id}-\Pi_n)L^2(\mathbb{R}/2\pi\mathbb{Z})$, has a bounded inverse $R^0_{\gamma,a,c,n}$ given by*

$$R^0_{\gamma,a,c,n}(f) = \sum_{p\neq n}\frac{\widehat{f}_p\mathrm{e}^{\mathrm{i}px}}{\mathrm{i}\omega^0_n(\gamma,a,c)-\mathrm{i}\omega^0_p(\gamma,a,c)},$$

where $f\in(\mathrm{id}-\Pi_n)L^2(\mathbb{R}/2\pi\mathbb{Z})$, and $\widehat{f}_p$ represent the Fourier coefficients of f, $f=\sum_{p\neq n}\widehat{f}_p\mathrm{e}^{\mathrm{i}px}$. Furthermore,

$$\|R^0_{\gamma,a,c,n}\|_{\mathcal{L}(H_n,V^1_n)} \leq \frac{M_0}{1+|n|}, \tag{4.2}$$

in which $V^1_n=(\mathrm{id}-\Pi_n)H^1(\mathbb{R}/2\pi\mathbb{Z})$ and

$$M_0 = \tfrac{1}{m_4}, \qquad \textit{under hypothesis } \text{(H1)},$$
$$M_0 = \tfrac{1}{m_5^*}, \qquad \textit{under hypothesis } \text{(H2)}.$$

Here m_4 and m_5^* are the constants defined in Lemma 4.2.

The resolvent estimate (4.2) guarantees robustness of spectral gaps under perturbations with growth $\mathrm{O}(n)$ in the Fourier mode, which is precisely the type of perturbation encountered by the derivative $kp_{a,c}\partial_x u$. It is a direct consequence of the spectral gaps between large eigenvalues provided in Lemma 4.2, and the central (and basically only) ingredient to the proof of absence of unstable eigenvalues outside a neighborhood of the origin.

The rest of this section is occupied by the proof of Lemma 4.2.

Proof. [of Lemma 4.2.] We first compute a lower bound for $|\omega^0_n(\gamma,a,c)-\omega^0_p(\gamma,a,c)|$, for $n,p\in\mathbb{Z}$, $n\neq p$, and $|n|\geq 3$ or $|p|\geq 3$. We set $N=n+\gamma$, $P=p+\gamma$, and $R^2=N^2+P^2$. We then have

$$|\omega^0_n(\gamma,a,c)-\omega^0_p(\gamma,a,c)| = k^5|n-p|\left|\Delta_5-\tfrac{1}{k^2}\Delta_3-\tfrac{c}{k^4}\right|,$$

where

$$\begin{aligned}\Delta_5 = N^4 + P^4 + N^2P^2 + NP(N^2+P^2) \quad &\geq N^4 + P^4 + N^2P^2 - \frac{(N^2+P^2)^2}{2} \\ &= \frac{N^4+P^4}{2} \geq \frac{(N^2+P^2)^2}{4} = \frac{R^4}{4},\end{aligned}$$

and

$$0 < \Delta_3 = N^2 + P^2 + NP \leq N^2 + P^2 + \frac{N^2+P^2}{2} = \frac{3}{2}R^2.$$

Since $|n-p| \geq 1$, $k = 1 + \mathcal{O}(c)$ by (2.4), and since for $|n| \geq 3$ or $|p| \geq 3$, and any $\gamma \in [-\frac{1}{2}, \frac{1}{2})$, we have $R^2 \geq \frac{25}{4}$, we obtain

$$\begin{aligned}|\omega_n^0(\gamma,a,c) - \omega_p^0(\gamma,a,c)| \quad &\geq k^5R^4\left(\frac{1}{4} - \frac{3}{2}\frac{1}{k^2R^2} - \frac{|c|}{k^4R^4}\right) \\ &\geq k^5R^4\left(\frac{1}{4} - \frac{6}{25k^2} - \frac{16|c|}{25^2k^4}\right) \geq \frac{1}{200}R^4,\end{aligned} \tag{4.3}$$

for $|c|$ sufficiently small, and any $a \in (-a_0, a_0)$. Moreover,

$$1 + |N| = 1 + |n+\gamma| \geq 1 + |n| - |\gamma| \geq \frac{1}{2} + |n| \geq \frac{1+|n|}{2},$$

so that

$$\frac{(1+|n|)(1+|p|)}{4} \leq (1+|N|)(1+|P|) \leq (2+N^2+P^2) = 2 + R^2. \tag{4.4}$$

Combining (4.3) and (4.4) we obtain,

$$\frac{|\omega_n^0(\gamma,a,c) - \omega_p^0(\gamma,a,c)|}{(1+|n|)(1+|p|)} \geq \frac{1}{800}\,\frac{R^4}{2+R^2} \geq m_4 > 0, \tag{4.5}$$

for some positive constant m_4. This inequality holds for $|c|$ sufficiently small, $a \in (-a_0, a_0)$, $\gamma \in [-\frac{1}{2}, \frac{1}{2})$, and $n, p \in \mathbb{Z}$ with $n \neq p$, and $|n| \geq 3$ or $|p| \geq 3$.

To finish the proof of the lemma it remains to find a positive lower bound for the modulus of $\Omega_{n,p}(\gamma,a,c) := \omega_n^0(\gamma,a,c) - \omega_p^0(\gamma,a,c)$ for $|n| \leq 2$, $|p| \leq 2$, and $n \neq p$. Observing that

$$\Omega_{n,p}(\gamma,a,c) = -\Omega_{p,n}(\gamma,a,c) = -\Omega_{-n,-p}(-\gamma,a,c), \tag{4.6}$$

we only have to consider $\Omega_{1,0}$, $\Omega_{1,-1}$, and $\Omega_{2,p}$ for $-2 \leq p \leq 1$. We give the calculations for $\Omega_{2,1}$ and $\Omega_{1,0}$, the other quantities can be treated in a similar way.

For $\Omega_{2,1}$, we set $x = 1+\gamma \geq \frac{1}{2}$, and find, for $|c|$ sufficiently small, $a \in (-a_0, a_0)$, and $\gamma \in [-\frac{1}{2}, \frac{1}{2})$ that

$$\begin{aligned}\Omega_{2,1}(\gamma,a,c) \quad &= k^5(5x^4 + 10x^3 + 10x^2 + 5x + 1) - k^3(3x^2 + 3x + 1) - ck \\ &= 5k^5x^4 + 10k^5x^3 + k^3(10k^2 - 3)x^2 + k^3(5k^2 - 3)x + k^5 - k^3 - ck \\ &\geq \tfrac{7}{2}x^2 + x - \tfrac{1}{2} \geq \tfrac{7}{8} > 0,\end{aligned}$$

where we have used the fact that $k = 1 + \mathcal{O}(c)$.

For $\Omega_{1,0}$ we only need a lower bound for $|\gamma| \geq \gamma^*$. We have

$$\Omega_{1,0}(\gamma, a, c) = 5k^5\gamma^4 + 10k^5\gamma^3 + \gamma^2(10k^5 - 3k^3) + k^3\gamma(5k^2 - 3) + k^5 - k^3 - ck,$$

and

$$|\Omega_{1,0}(\gamma, a, 0)| = |5\gamma^4 + 10\gamma^3 + 7\gamma^2 + \gamma| \geq \frac{3}{8}|\gamma|,$$

for $\gamma \in [-\frac{1}{2}, \frac{1}{2})$. Since

$$|\Omega_{1,0}(\gamma, a, c) - \Omega_{1,0}(\gamma, a, 0)| \leq \sup_{|a| \leq a_0, |\gamma| \leq \frac{1}{2}, |c| \leq c_0} \left| \frac{\partial \Omega_{1,0}}{\partial c}(\gamma, a, c) \right| |c| =: M|c|,$$

we find

$$|\Omega_{1,0}(\gamma, a, c)| \geq |\Omega_{1,0}(\gamma, a, 0)| - |\Omega_{1,0}(\gamma, a, c) - \Omega_{1,0}(\gamma, a, 0)| \geq \frac{3}{8}|\gamma| - M|c| \geq m_5^*\gamma^*,$$

for any $|\gamma| \geq \gamma^*$, $a \in (-a_0, a_0)$, $c \in (-c^*, c^*)$ with $c^* = c^*(\gamma^*)$ sufficiently small. ∎

5 Spectrum for γ bounded away from zero

This section is devoted to the proof of Proposition 3.4 (i): we show that for γ bounded away from zero the spectrum of $L_{\gamma,a,c}$ is contained in the imaginary axis. This result is proved in two steps. First, we locate the spectrum of $L_{\gamma,a,c}$ in a neighborhood of the spectrum of $L^0_{\gamma,a,c}$ (Lemma 5.1), and then show the absence of spectrum outside of this neighborhood (Lemma 5.2). Notice that for these values of γ the spectrum of $L^0_{\gamma,a,c}$ consists only of simple eigenvalues. However, we cannot use classical perturbation results, since we have here infinitely many eigenvalues.

Notation: We write $B(z, r)$ for the ball of radius r centered at z in the complex plane.

Lemma 5.1 *For every $\gamma^* \in (0, \frac{1}{2})$ there exist $c_6^* > 0$ and $r_6^* > 0$ such that for every $c \in (-c_6^*, c_6^*)$, $a \in (-a_0, a_0)$, $\gamma \in [-\frac{1}{2}, \frac{1}{2})$ with $|\gamma| \geq \gamma^*$, and every $n \in \mathbb{Z}$, the operator $L_{\gamma,a,c}$ has a unique eigenvalue $\mathrm{i}\omega_n(\gamma, a, c)$ in $B\big(\mathrm{i}\omega_n^0(\gamma, a, c), r_6^*(1 + |n|)\big)$ and this eigenvalue is purely imaginary. Moreover, the map $(a, c) \mapsto \omega_n(\gamma, a, c)$ is analytic, and $\omega_n(\gamma, 0, 0) = \omega_n^0(\gamma, 0, 0) = (n + \gamma)^5 - (n + \gamma)^3$.*

Lemma 5.2 *For every $\gamma^* \in (0, \frac{1}{2})$ there exist $c_7^* > 0$ such that for every $c \in (-c_7^*, c_7^*)$, $a \in (-a_0, a_0)$, $\gamma \in [-\frac{1}{2}, \frac{1}{2})$ with $|\gamma| \geq \gamma^*$, the spectrum of $L_{\gamma,a,c}$ satisfies*

$$\mathrm{spec}(L_{\gamma,a,c}) \subset \bigcup_{n \in \mathbb{Z}} B\big(\mathrm{i}\omega_n^0(\gamma, a, c), r_6^*(1 + |n|)\big),$$

in which r_6^* is the constant given in Lemma 5.1. Consequently, $\mathrm{spec}(L_{\gamma,a,c}) \subset \mathrm{i}\mathbb{R}$.

Proposition 3.4 (i) is a direct consequence of Lemma 5.1 and 5.2. The remainder of this section is occupied by the proofs of these two lemmas. The key point in Lemma 5.1 is that $c_6^* > 0$ and $r_6^* > 0$ only depend upon γ^* and not upon n.

Proof. [of Lemma 5.1] We fix $\gamma^* \in (0, \frac{1}{2})$, $\gamma \in [-\frac{1}{2}, \frac{1}{2})$ with $|\gamma| \geq \gamma^*$, $a \in (-a_0, a_0)$ and $n \in \mathbb{Z}$. Then $\mathrm{i}\omega_n^0(\gamma, a, c)$ has the properties stated in Corollary 4.3. In particular, the restriction of $\mathrm{i}\omega_n^0(\gamma, a, c)\ \mathrm{id} - L^0_{\gamma,a,c}$ to $H_n = (\mathrm{id} - \Pi_n)L^2(\mathbb{R}/2\pi\mathbb{Z})$ has a bounded inverse $R^0_{\gamma,a,c,n}$, with a bound in $\mathcal{L}(H_n, V_n^1)$ given by $\frac{1}{m_5^*(1+|n|)}$, where $V_n^1 = (\mathrm{id} - \Pi_n)H^1(\mathbb{R}/2\pi\mathbb{Z})$. We denote by $u_n(x) = \mathrm{e}^{\mathrm{i}nx}$ the eigenvector associated with the eigenvalue $\mathrm{i}\omega_n^0(\gamma, a, c)$ of $L^0_{\gamma,a,c}$.

In order to determine the eigenvalues of $L_{\gamma,a,c}$ in some neighborhood $B\big(\mathrm{i}\omega_n^0(\gamma, a, c), r^*(1+|n|)\big)$ of the eigenvalue $\mathrm{i}\omega_n^0(\gamma, a, c)$, we have to find pairs $(\lambda, u) \in \mathbb{C} \times H^5(\mathbb{R}/2\pi\mathbb{Z})$, with $u \not\equiv 0$, $|\lambda - \mathrm{i}\omega_n^0(\gamma, a, c)| < r^*\ (1 + |n|)$, and

$$L_{\gamma,a,c}u - \lambda u = 0. \tag{5.1}$$

The key point of this analysis is that we want to solve this equation for r^* and $|c| < c^*$ sufficiently small depending only upon γ^* and, in particular, not upon n.

Notice that $(\mathrm{i}\omega_n^0(\gamma, a, 0), u_n)$ verifies (5.1) at $c = 0$, since $L_{\gamma,a,0} = L^0_{\gamma,a,0}$. We therefore seek (λ, u) in the form

$$\lambda = \mathrm{i}\omega_n^0(\gamma, a, c) + (1 + |n|)\mu, \quad u = \alpha u_n + v, \tag{5.2}$$

in which $\mu, \alpha \in \mathbb{C}$ and $\Pi_n v = 0$. By substituting (5.2) into (5.1) we find the equivalent system

$$\Pi_n L^1_{\gamma,a,c}(\alpha u_n + v) - \mu(1 + |n|)\alpha u_n = 0$$
$$(L^0_{\gamma,a,c} - \mathrm{i}\omega_n^0(\gamma, a, c))v + (\mathrm{id} - \Pi_n)L^1_{\gamma,a,c}(\alpha u_n + v) - \mu(1 + |n|)v = 0.$$

Here, the linear operator $L^1_{\gamma,a,c}$ is a relatively bounded perturbation of $L^0_{\gamma,a,c}$, with norm

$$\|L^1_{\gamma,a,c}\|_{\mathcal{L}(H^1,L^2)} \leq M_1|c|, \tag{5.3}$$

for some positive constant M_1, and any $\gamma \in [-\frac{1}{2}, \frac{1}{2})$, $a \in (-a_0, a_0)$, and $c \in (-c_0, c_0)$. Since the restriction of $\mathrm{i}\omega_n^0(\gamma, a, c)\mathrm{id} - L^0_{\gamma,a,c}$ to H_n is invertible we can rewrite the system above as

$$\alpha\mu = \frac{1}{1+|n|}\pi_n L^1_{\gamma,a,c}v + \frac{1}{1+|n|}\pi_n L^1_{\gamma,a,c}(\alpha u_n) \tag{5.4}$$
$$v = R^0_{\gamma,a,c,n}\left((\mathrm{id} - \Pi_n)L^1_{\gamma,a,c} - (1 + |n|)\mu\right)v + R^0_{\gamma,a,c,n}(\mathrm{id} - \Pi_n)L^1_{\gamma,a,c}(\alpha u_n), \tag{5.5}$$

in which $R^0_{\gamma,a,c,n}$ is given by Corollary 4.3 (ii), and where for $f \in L_2(\mathbb{R}/2\pi\mathbb{Z})$, $\pi_n(f) = \widehat{f}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\mathrm{e}^{-\mathrm{i}ns}ds$.

We claim that $\alpha \neq 0$. Indeed, if $\alpha = 0$, we deduce from (5.5), (5.3) and (4.2) that

$$\|v\|_{H^1} \leq \frac{1}{m_5^*}\left(\frac{M_1|c|}{1+|n|} + |\mu|\right)\|v\|_{H^1} \leq \frac{1}{m_5^*}\left(M_1|c| + |\mu|\right)\|v\|_{H^1}.$$

Hence, for $|c| < \frac{m_5^*}{2M_1}$ and $|\mu| < \frac{m_5^*}{2}$ the unique solution of (5.4), (5.5) is $v = 0$ which yields a solution $u = 0$ to (5.1).

Since $\alpha \neq 0$ and the equation (5.1) is linear, we may assume without loss of generality that $\alpha = 1$. Then, (5.4), (5.5) read

$$(\mu, v) = \mathcal{N}_{n,\gamma,a}(\mu, v; c), \tag{5.6}$$

where $\mathcal{N}_{n,\gamma,a}$ is analytic from $\mathbb{C}\times H^1(\mathbb{R}/2\pi\mathbb{Z})\times(-c_5^*, c_5^*)$ into $\mathbb{C}\times H^5(\mathbb{R}/2\pi\mathbb{Z}) \subset \mathbb{C}\times H^1(\mathbb{R}/2\pi\mathbb{Z})$. In particular, if $(\mu, v) \in \mathbb{C} \times H^1(\mathbb{R}/2\pi\mathbb{Z})$ satisfies (5.6), then v lies in $H^5(\mathbb{R}/2\pi\mathbb{Z})$. Thus we can only look for solutions (μ, v) of (5.6) in $\mathbb{C} \times H^1(\mathbb{R}/2\pi\mathbb{Z})$. Here μ may be chosen small, $|\mu| < r^*$, but not v which belongs to the entire space $H^1(\mathbb{R}/2\pi\mathbb{Z})$. However, if

$$|c| \leq \frac{m_5^*}{4M_1}, \quad |\mu| \leq \frac{m_5^*}{4},$$

we can solve (5.5) for v, $v = v(\mu)$, and find an a priori bound

$$\|v\|_{H^1} \leq \frac{2M_1|c|}{m_5^*}.$$

Consequently, it suffices to solve (5.6) for

$$(\mu, v) \in \mathcal{U}^* := \overline{B}\left(0, \frac{m_5^*}{4}\right) \times \overline{B}\left(0, \frac{2M_1c^*}{m_5^*}\right) \subset \mathbb{C} \times H^1(\mathbb{R}/2\pi\mathbb{Z}),$$

and sufficiently small $c \in (-c^*, c^*)$. We do this by a fixed point argument. Indeed, a direct calculation using the estimates for the linear operators $R^0_{\gamma,a,c,n}$ and $L^1_{\gamma,a,c}$, shows that $\mathcal{N}_{n,\gamma,a}(\cdot,\cdot;c) : \mathcal{U}^* \to \mathcal{U}^*$ is well defined for c^* sufficiently small. In addition, for any (μ_1, v_1) and (μ_2, v_2) in $\mathcal{U}^*$, we find

$$\begin{aligned}
\|\mathcal{N}_{n,\gamma,a}(\mu_1, v_1; c) - \mathcal{N}_{n,\gamma,a}(\mu_2, v_2; c)\|_{\mathbb{C}\times H^1} &\leq M_1|c|\|v_1 - v_2\|_{H^1} + \frac{M_1|c|}{m_5^*}\|v_1 - v_2\|_{H^1} \\
&\quad + \frac{1}{m_5^*}\|v_1\|_{H^1}\,|\mu_1 - \mu_2| + \frac{1}{m_5^*}|\mu_2|\|v_1 - v_2\|_{H^1} \\
&\leq \left(1 + \frac{1}{m_5^*}\right) M_1c^*\|v_1 - v_2\|_{H^1} + \frac{1}{4}\|v_1 - v_2\|_{H^1} + \frac{2M_1c^*}{(m_5^*)^2}|\mu_1 - \mu_2|,
\end{aligned}$$

so that $\mathcal{N}_{n,\gamma,a}(\cdot,\cdot;c)$ is a contraction on $\mathcal{U}^*$ provided c^* is small enough.

From the arguments above, we conclude that there exist c_6^* sufficiently small, $r_6^* = \frac{m_5^*}{4}$, such that for any $c \in (-c_6^*, c_6^*)$, $a \in (-a_0, a_0)$, $\gamma \in [-\frac{1}{2}, \frac{1}{2})$ with $|\gamma| \geq \gamma^*$, and any $n \in \mathbb{Z}$, the system (5.6) admits a unique solution $(\mu, v) := \big(\mu_n^1(\gamma, a, c), v_n^1(\gamma, a, c)\big)$ in $B(0, r_6^*) \times H^1(\mathbb{R}/2\pi\mathbb{Z})$. This solution yields a unique eigenvalue $\mathrm{i}\omega_n^0(\gamma, a, c)+\mu_n^1(\gamma, a, c)$ of $L_{\gamma,a,c}$ in $B(\mathrm{i}\omega_n^0(\gamma, a, c), r_6^*(1+|n|))$. Finally, the symmetry of the spectrum of $L_{\gamma,a,c}$ (see Remark 3.5 (i)) implies that this eigenvalue is purely imaginary. ∎

Proof. [of Lemma 5.2] As in the proof of Lemma 5.1, we write $L_{\gamma,a,c} = L^0_{\gamma,a,c} + L^1_{\gamma,a,c}$. Assume that $\lambda \notin \bigcup_{n\in\mathbb{Z}} B\big(\mathrm{i}\omega^0_n(\gamma,a,c), r^*_6(1+|n|)\big)$. Then

$$|\lambda - \mathrm{i}\omega_n(\gamma)| \geq r^*_6(1+|n|),$$

for every $n \in \mathbb{Z}$, and Lemma 4.1 (v) ensures that $\lambda\mathrm{id} - L^0_{\gamma,a,c}$ is invertible with

$$\|(\lambda\mathrm{id} - \mathrm{L}^0_{\gamma,\mathrm{a},\mathrm{c}})^{-1}\|_{\mathcal{L}(L^2,H^1)} \leq \frac{1}{r^*_6}.$$

This estimate and the inequality (5.3) imply that $\lambda\mathrm{id} - L_{\gamma,a,c}$ is invertible for sufficiently small $c \in (-c^*_7, c^*_7)$, and the lemma is proved. Notice that the constant c^*_7 only depends upon γ^* and does not depend upon a, λ and γ with $|\gamma| \geq \gamma^*$, since r^*_6 and the constant M_1 in (5.3) are independent of γ, a, c, and λ. ∎

6 Spectrum for γ close to zero

This section is devoted to the proof of Proposition 3.4 (ii): we prove that for γ close to zero the spectrum of $L_{\gamma,a,c}$ is included in the imaginary axis. We use again a perturbation argument, by regarding $L_{\gamma,a,c}$ as a perturbation of $L^0_{\gamma,a,c}$ by $L^1_{\gamma,a,c}$. The main difference between $|\gamma|$ small and γ bounded away from zero is that $L^0_{\gamma,a,c}$ has only simple isolated eigenvalues in the second case whereas in the first case, $L^0_{\gamma,a,c}$ has three eigenvalues which are arbitrary close and which coincide at $\gamma = 0$. In order to determine the spectrum of $L_{\gamma,a,c}$ for $|\gamma|$ small, we proceed in three steps. We first show in Section 6.1 that for $|\gamma|$ small, the spectrum of $L_{\gamma,a,c}$ separates into two parts,

$$\mathrm{spec}(L_{\gamma,a,c}) = \sigma_1 \cup \sigma_2, \qquad \sigma_1 \subset B(0,1), \qquad \sigma_2 \subset \mathbb{C} \setminus \overline{B(0,1)}.$$

Next, we analyze σ_2 in Section 6.2. We argue as in Section 5 for γ bounded away from zero, and show that σ_2 consists of simple, isolated eigenvalues which are all purely imaginary. Finally, in Section 6.3, a careful bifurcation analysis enables us to determine σ_1 by unfolding the triple zero eigenvalue of $L^0_{\gamma,a,c}$ at $\gamma = 0$.

6.1 Separation of spectrum

From the dispersion relation (4.1) we deduce the following separation property for the spectrum of $L^0_{\gamma,a,c}$.

Lemma 6.1 *There exist positive constants c_8 and γ_8 such that for any $c \in (-c_8, c_8)$, $a \in (-a_0, a_0)$, and $\gamma \in (-\gamma_8, \gamma_8)$, the eigenvalues $\mathrm{i}\omega^0_n = \mathrm{i}\omega^0_n(\gamma,a,c)$ of $L^0_{\gamma,a,c}$ satisfy*

$$\{\mathrm{i}\omega^0_0, \mathrm{i}\omega^0_1, \mathrm{i}\omega^0_{-1}\} \subset B(0,\tfrac{1}{2}), \quad \{\mathrm{i}\omega^0_n\,;\ n\in\mathbb{Z},\ |n|\geq 2\} \subset \mathbb{C}\backslash B(0,2).$$

This lemma provides us with a spectral decomposition through the circle $C(0,1)$. Inside this circle we find the three eigenvalues $\mathrm{i}\omega_0^0$, $\mathrm{i}\omega_1^0$, $\mathrm{i}\omega_{-1}^0$, the remaining eigenvalues being simple and located outside this circle. We denote by E the invariant subspace spanned by the eigenvectors 1, $\mathrm{e}^{\mathrm{i}x}$, and $\mathrm{e}^{-\mathrm{i}x}$, associated to the three eigenvalues $\mathrm{i}\omega_0^0$, $\mathrm{i}\omega_1^0$, and $\mathrm{i}\omega_{-1}^0$, respectively. Let $\Pi^0 : L^2(\mathbb{R}/2\pi\mathbb{Z}) \to E$ be the corresponding spectral projection,

$$\Pi^0(f) = \pi_0(f) + \pi_{+1}(f)\mathrm{e}^{\mathrm{i}x} + \pi_{-1}(f)\mathrm{e}^{-\mathrm{i}x},$$

in which

$$\pi_0(f) = \widehat{f}_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \quad \pi_{\pm 1}(f) = \widehat{f}_{\pm 1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mp \mathrm{i}x}\,\mathrm{d}x.$$

We also consider the spaces $H = (\mathrm{id}-\Pi^0)L^2(\mathbb{R}/2\pi\mathbb{Z})$ and $V = H \cap H^5(\mathbb{R}/2\pi\mathbb{Z})$, and decompose the resolvent operator,

$$R^0_{\gamma,a,c}(\lambda) = R^0(\lambda) + \widetilde{R}^0_{\gamma,a,c}(\lambda), \tag{6.1}$$

in which

$$R^0(\lambda)(f) = \sum_{n=0,\pm1} \frac{\widehat{f}_n \mathrm{e}^{\mathrm{i}nx}}{\lambda - \mathrm{i}\omega_n^0(\gamma,a,c)}, \quad \widetilde{R}^0_{\gamma,a,c}(\lambda)(f) = \sum_{|n|\geq 2} \frac{\widehat{f}_n \mathrm{e}^{\mathrm{i}nx}}{\lambda - \mathrm{i}\omega_n^0(\gamma,a,c)}.$$

Notice that the space E and the projection Π^0 do not depend upon γ, a, and c.

We now turn back to the operator $L_{\gamma,a,c}$. This operator is a relatively bounded perturbation of $L^0_{\gamma,a,c}$ with small relative bound $\mathrm{O}(c)$. Classical perturbation theory for linear operators shows that the three eigenvalues of $L^0_{\gamma,a,c}$ inside the circle $C(0,1)$ persist as the only spectral values of $L_{\gamma,a,c}$ inside this circle. More precisely, we have the following result.

Lemma 6.2 *There exists a positive constant $c_9 \leq c_8$, such that for any $c \in (-c_9, c_9)$, $a \in (-a_0, a_0)$, and $\gamma \in (-\gamma_8, \gamma_8)$, the linear operator $L_{\gamma,a,c}$ has the following properties.*

(i) *The circle $C(0,1)$ separates the spectrum of $L_{\gamma,a,c}$ into two disjoint parts,*

$$\mathrm{spec}(L_{\gamma,a,c}) = \sigma_1 \cup \sigma_2, \qquad \sigma_1 \subset B(0,1), \qquad \sigma_2 \subset \mathbb{C} \setminus \overline{B(0,1)}.$$

Inside this circle there are exactly three eigenvalues, counted with multiplicities.

(ii) *The three dimensional space $E_{\gamma,a,c}$ associated to the three spectral values inside the circle $C(0,1)$, and the corresponding spectral projection $\Pi_{\gamma,a,c}$ are analytic in γ, a and c.*

(iii) *The three eigenvalues inside the circle $C(0,1)$ are the solutions λ of the equation*

$$\det\left(\lambda\Pi_{\gamma,a,c} - \Pi_{\gamma,a,c}L_{\gamma,a,c}\Pi_{\gamma,a,c}\right) = 0, \tag{6.2}$$

where the operator in the parentheses is considered as linear map from $E_{\gamma,a,c}$ into itself.

(iv) *The projection* $\Pi_{\gamma,a,c}$ *satisfies the equality*

$$\Pi_{\gamma,a,c} = \Pi^0 + \frac{1}{2\pi \mathrm{i}} \int_{C(0,1)} \sum_{\ell \geq 1} R^0_{\gamma,a,c}(\lambda) \left(L^1_{\gamma,a,c} R^0_{\gamma,a,c}(\lambda) \right)^{\ell} \mathrm{d}\lambda, \tag{6.3}$$

in which $R^0_{\gamma,a,c}(\lambda)$ *is the resolvent of* $L^0_{\gamma,a,c}$*. Hence,* $\Pi_{\gamma,a,c}$ *is analytic with respect to* (γ, a, c)*.*

Proof. The first three statements follow from the results in [7, Chapter IV, § 3.5]. Finally, the last statement is obtained from the formula for the resolvent $R_{\gamma,a,c}(\lambda)$ of $L_{\gamma,a,c}$,

$$R_{\gamma,a,c}(\lambda) = R^0_{\gamma,a,c}(\lambda) \left(\mathrm{id} - L^1_{\gamma,a,c} R^0_{\gamma,a,c}(\lambda) \right)^{-1} = R^0_{\gamma,a,c}(\lambda) + \sum_{\ell \geq 1} R^0_{\gamma,a,c}(\lambda) \left(L^1_{\gamma,a,c} R^0_{\gamma,a,c}(\lambda) \right)^{\ell},$$

and the Dunford integral formula for the projection. ∎

6.2 Spectrum of $\mathbf{L}_{\gamma,\mathbf{a},\mathbf{c}}$ outside $\mathbf{B}(0,1)$

In this section we prove that the spectrum σ_2 of $L_{\gamma,a,c}$ outside $B(0,1)$ is included in the imaginary axis. This part of the spectrum of $L_{\gamma,a,c}$ is located in a neighborhood of the eigenvalues $\mathrm{i}\omega^0_n(\gamma, a, c)$, $|n| \geq 2$, of $L^0_{\gamma,a,c}$. For sufficiently small c these eigenvalues satisfy Hypothesis H1 in Corollary 4.3, and therefore share spectral gap properties with the eigenvalues for γ bounded away from zero. Consequently, in order to locate σ_2, we can proceed as in Section 5 for γ bounded away from zero. We have the following two results.

Lemma 6.3 *There exist positive constants* $c_{10} \leq c_9$ *and* r_{10} *such that for any* $c \in (-c_{10}, c_{10})$*,* $a \in (-a_0, a_0)$*,* $\gamma \in (-\gamma_8, \gamma_8)$*, and any* $n \in \mathbb{Z}$ *with* $|n| \geq 2$*, the operator* $L_{\gamma,a,c}$ *has a unique eigenvalue* $\mathrm{i}\omega_n(\gamma, a, c)$ *in* $B\big(\mathrm{i}\omega^0_n(\gamma, a, c), r_{10}(1 + |n|)\big)$ *and this eigenvalue is purely imaginary. Moreover, the map* $(a, c) \mapsto \omega_n(\gamma, a, c)$ *is analytic, and* $\omega_n(\gamma, 0, 0) = \omega^0_n(\gamma, 0, 0) = (n + \gamma)^5 - (n + \gamma)^3$*.*

Lemma 6.4 *There exists a positive constant* $c_{11} \leq c_{10}$ *a such that for any* $c \in (-c_{11}, c_{11})$*,* $a \in (-a_0, a_0)$*, and* $\gamma \in (-\gamma_8, \gamma_8)$*, the spectrum of* $L_{\gamma,a,c}$ *satisfies*

$$\sigma_2 = \mathrm{spec}(L_{\gamma,a,c}) \cap \left\{ \mathbb{C} \setminus \overline{B(0,1)} \right\} \subset \bigcup_{|n| \geq 2} B\big(\mathrm{i}\omega^0_n(\gamma, a, c), r_{10}(1 + |n|)\big).$$

Consequently, $\sigma_2 \subset \mathrm{i}\mathbb{R}$*.*

The proofs of these two results are very similar to the ones of Lemma 5.1 and Lemma 5.2 in Section 5, and we therefore omit the details. They both rely on the resolvent estimate (4.2), in which we take $M_0 = \frac{1}{m_4}$ here, while $M_0 = \frac{1}{m_5^*}$ was used in Section 5.

6.3 Spectrum of $\mathbf{L}_{\gamma,\mathbf{a},\mathbf{c}}$ inside $\mathbf{B}(\mathbf{0},\mathbf{1})$

This last section is devoted to the study of the spectrum of $L_{\gamma,a,c}$ inside the circle $C(0,1)$. This part of the spectrum consists of precisely three eigenvalues counted with multiplicities. These eigenvalues are solutions of the equation (6.2), and they can be located provided one can compute and solve (6.2). This turns out to be a rather difficult task, in particular due to the fact that the range $E_{\gamma,a,c}$ of the projection $\Pi_{\gamma,a,c}$ is a three dimensional space depending upon the parameters γ, a, and c. Such a direct calculation can be performed for the (much simpler) KdV equation (see Remark 3.1), using the Taylor expansions in the *two* parameters γ and a of the different quantities involved in (6.2). The presence of a third parameter c, and the more complicated structure of the Kawahara equation make this calculation much harder. Therefore, we choose here a slightly different approach. Instead of working in the parameter dependent space $E_{\gamma,a,c}$, we work in the fixed space E spanned by the functions 1, $e^{\mathrm{i}x}$, and $e^{-\mathrm{i}x}$. In other words, we replace the spectral subspace $E_{\gamma,a,c}$ associated to $L_{\gamma,a,c}$ by the spectral subspace E to $L^0_{\gamma,a,c}$. This can be achieved via the next lemma.

Lemma 6.5 *Consider the spectral subspace E to $L^0_{\gamma,a,c}$ spanned by the eigenvectors 1, $\mathrm{e}^{\mathrm{i}x}$, and $\mathrm{e}^{-\mathrm{i}x}$, associated to the three eigenvalues $\mathrm{i}\omega^0_0$, $\mathrm{i}\omega^0_1$, and $\mathrm{i}\omega^0_{-1}$, respectively, and Π^0 the corresponding spectral projection, defined in Section 6.1. There exists a positive constant $c_{12} \leq c_{11}$ such that for every $c \in (-c_{12}, c_{12})$, $a \in (-a_0, a_0)$, and $\gamma \in (-\gamma_8, \gamma_8)$, the projections $\Pi_{\gamma,a,c}$ and Π^0 realize isomorphisms between $E \to E_{\gamma,a,c}$ and $E_{\gamma,a,c} \to E$, respectively. Consequently, the three eigenvalues of $L_{\gamma,a,c}$ inside the circle $C(0,1)$ are the solutions λ of the equation*

$$\det\left(\Pi^0(\lambda\Pi_{\gamma,a,c} - L_{\gamma,a,c}\Pi_{\gamma,a,c})\Pi^0\right) = 0, \tag{6.4}$$

where again the operator in the parentheses is considered as a map from E_0 to E_0.

Proof. The two projections $\Pi_{\gamma,a,c}$ and Π^0 have finite-dimensional range and verify

$$\begin{aligned}\|\Pi_{\gamma,a,c} - \Pi^0\| &= \left\|\frac{1}{2\pi\mathrm{i}}\int_{C(0,1)}\sum_{n\geq 1} R^0_{\gamma,a,c}(\lambda)\left(L^1_{\gamma,a,c}R^0_{\gamma,a,c}(\lambda)\right)^n \mathrm{d}\lambda\right\| \\ &\leq M|c| < \min\left(\frac{1}{\|\Pi_{\gamma,a,c}\|}, \frac{1}{\|\Pi^0\|}\right),\end{aligned}$$

for sufficiently small $|c|$, and some positive constant M, since on the circle $C(0,1)$ the resolvent $R^0_{\gamma,a,c}(\lambda)$ is uniformly bounded (see Lemma 4.1 (v) and Lemma 6.1), and since $\|L^1_{\gamma,a,c}\| = \mathrm{O}(|c|)$ (see equation (5.3)). This inequality together with the result in Appendix B, imply that $\Pi_{\gamma,a,c}$ and Π^0 realize isomorphisms between their images, and the lemma is proved. ∎

We now use (6.4) to locate the position of the three eigenvalues of $L_{\gamma,a,c}$ inside the circle $C(0,1)$. Notice that the right hand side of this formula is a polynomial in λ of degree 3, with coefficients depending analytically upon γ, a, and c (see Lemma 6.2 (iv)). We denote this polynomial by $D(\gamma,a,c)[\lambda]$, so that (6.4) reads

$$D(\gamma,a,c)[\lambda] := \det\left(\Pi^0(\lambda - L_{\gamma,a,c})\Pi_{\gamma,a,c}\Pi^0\right) = 0. \tag{6.5}$$

Remark 6.6 *From the symmetry properties of the linear operators $L_{\gamma,a,c}$ we deduce that if $D(\gamma,a,c)[\lambda]=0$, then*

$$D(\gamma,a,c)[-\overline{\lambda}]=0, \quad D(-\gamma,a,c)[-\lambda]=0, \quad D(-\gamma,a,c)[\overline{\lambda}]=0.$$

Lemma 6.7 *For every $c\in(-c_{12},c_{12})$, $a\in(-a_0,a_0)$, and $\gamma\in(-\gamma_8,\gamma_8)$, the polynomial $D(\gamma,a,c)[\lambda]$ has the following properties.*

(i) *The coefficients of D depend analytically upon (γ,a,c) in some neighborhood of 0 in $\mathbb{R}^3$.*

(ii) *$D(-\gamma,a,c)[\lambda] = -D(\gamma,a,c)[-\lambda]$ and $D(\gamma,-a,c)[\lambda]=D(\gamma,a,c)[\lambda]$.*

(iii) *At $a=0$, the three roots of $D(\gamma,0,c)[\lambda]$ are $\mathrm{i}\omega_n^0(\gamma,0,c)$, $n=0,\pm1$.*

(iv) *At $\gamma=0$, zero is a triple root of $D(0,a,c)[\lambda]$, and $D(0,a,c)[\lambda]=\alpha\lambda^3$ with $\alpha=1+\mathrm{O}(a^2c)$.*

(v) *Set $\lambda=\mathrm{i}k\nu$ with $k=k(a,c)$ given in Theorem 1. Then*

$$D(\gamma,a,c)[\mathrm{i}k\nu] = (\mathrm{i}k)^3\alpha(\gamma,a,c)\Delta'(\gamma,a,c)[\nu],$$

where $\Delta'(\gamma,a,c)[\nu]$ is a polynomial in ν of degree 3 with real coefficients which depend analytically upon (γ,a,c) in some neighborhood of 0 in $\mathbb{R}^3$. Moreover, $\alpha(\gamma,a,c)=1+\mathrm{O}(a^2|c|+|\gamma|)$.

(vi) *There exist positive constants $c_{13}\le c_{12}$, a_{13}, and $\gamma_{13}\le\gamma_8$, such that for any $c\in(-c_{13},c_{13})$, $a\in(-a_{13}|c|^{1/4},a_{13}|c|^{1/4})$, and $\gamma\in(-\gamma_{13},\gamma_{13})$, the polynomial $\Delta'(\gamma,a,c)[\nu]$ has three real roots. Consequently, the three roots of the polynomial $D(\gamma,a,c)[\lambda]$ are purely imaginary.*

The last part of this lemma shows that the three eigenvalues of $L_{\gamma,a,c}$ inside $C(0,1)$ are purely imaginary. Consequently, the spectrum of $L_{\gamma,a,c}$ in a neighborhood of the origin is purely imaginary. This observation concludes the proof of Theorem 2.

Remark 6.8 *A direct calculation using the explicit formulae (6.5), (6.3), and (6.1) shows that in fact the coefficient $\alpha(\gamma,a,c)$ is also real. However, we do not need this fact in the proof of the lemma.*

Remark 6.9 *The regime where $\lambda, \gamma \sim 0$ is usually referred to as the long-wavelength, small frequency regime, where modulation equations can be derived. Our results so far show that, on the linear level, all instabilities necessarily derive from instabilities in this long-wavelength regime. A particular wedge of this long-wavelength regime is captured (formally) by the Witham equations [22, §16.15].*

Proof. [of Lemma 6.7] (i) Theorem 1 and Lemma 6.2 ensures that $\Pi_{\gamma,a,c}$ and $L_{\gamma,a,c}$ depend analytically on (γ, a, c). Hence this is also true for $D(\gamma, a, c)$.

(ii) Set $A_{\gamma,a,c}[\lambda] = \Pi^0(\lambda - L_{\gamma,a,c})\Pi_{\gamma,a,c}\Pi^0$, and we use the same notation for the 3×3 matrix associated to this operator in the basis $\{1, e^{ix}, e^{i-x}\}$. Since $L_{\gamma,a,c}$ anti-commutes with the reflection operator $Sf(x) = f(-x)$, $L_{-\gamma,a,c}S = -SL_{\gamma,a,c}$, we also have $\Pi_{-\gamma,a,c}S = S\Pi_{\gamma,a,c}$. Consequently $A_{-\gamma,a,c}[\lambda]S = -SA_{\gamma,a,c}[-\lambda]$, from which we deduce the first equality. Notice that the matrix associated with the reflector S in the basis above is

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.6}$$

The second equality is obtained in the same way by arguing with the translation $Tf(x) = f(x + \pi)$ instead of S (see Remark 2.1 (i)).

The statement (iii) is an immediate consequence of the fact that $L_{\gamma,0,c} = L^0_{\gamma,0,c}$.

For $a \neq 0$ and $c \neq 0$, the first part of (iv) follows from the Remark 3.5 (ii) which shows that zero is an eigenvalue of $L_{0,a,c}$ with algebraic multiplicity three, at least, and the fact that $L_{0,a,c}$ has exactly three eigenvalues inside the circle $C(0, 1)$. Then $D(0, a, c)[\lambda] = \alpha\lambda^3$, and a direct calculation using (6.5), (6.3), and (6.1) shows that $\alpha = 1 + \mathrm{O}(a^2c)$. For $a = 0$ or $c = 0$, (iv) is obtained by direct explicit computations since $L_{\gamma,0,c} = L^0_{\gamma,0,c}$ and $L_{\gamma,a,0} = L^0_{\gamma,a,0}$.

(v) We set $\lambda = ik\nu$ with $k = k(a, c)$ given in Theorem 1. Since k is close to 1 and therefore different from 0, we can write $D(\gamma, a, c)[ik\nu] = (ik)^3\Delta(\gamma, a, c)[\nu]$, in which $\Delta(\gamma, a, c)[\nu]$ is a polynomial in ν of degree three,

$$\Delta(\gamma, a, c)[\nu] = a_\Delta\nu^3 + b_\Delta\nu^2 + c_\Delta\nu + d_\Delta. \tag{6.7}$$

Here, the coefficients a_Δ, b_Δ, c_Δ, and d_Δ depend analytically upon γ, a, c. The result in (iv) implies that $a_\Delta = 1 + \mathrm{O}(a^2|c| + |\gamma|)$, so that we can write

$$\Delta(\gamma, a, c)[\nu] = a_\Delta\Delta'(\gamma, a, c)[\nu], \quad \Delta'(\gamma, a, c)[\nu] = \nu^3 + b'_\Delta\nu^2 + c'_\Delta\nu + d'_\Delta. \tag{6.8}$$

For any root $\lambda = ik\nu$ of $D(\gamma, a, c)[\lambda]$, we have that ν is a root of $\Delta'(\gamma, a, c)[\nu]$, and from the Remark 6.6, we find that if ν is a root of $\Delta'(\gamma, a, c)[\nu]$ then $\bar{\nu}$ is a root of $\Delta'(\gamma, a, c)[\nu]$

as well. Since the coefficient of the leading term ν^3 in the polynomial $\Delta'(\gamma, a, c)[\nu]$ is 1, we conclude that this polynomial has real coefficients, and (v) is proved. Moreover, since k and the coefficients of D depend analytically on (γ, a, c) this is also true for the coefficients of Δ and Δ'.

(vi) From (ii) we find that the coefficients b_Δ and d_Δ of the polynomial $\Delta(\gamma, a, c)[\nu]$ are odd in γ, that a_Δ and c_Δ are even in γ, and that a_Δ, b_Δ, c_Δ, d_Δ are even in a. Therefore, b'_Δ and d'_Δ are odd in γ, c'_Δ is even in γ, and b'_Δ, c'_Δ, d'_Δ are even in a. Next, from (iv), we have that $b'_\Delta = c'_\Delta = d'_\Delta = 0$ at $\gamma = 0$, so that we can write

$$b'_\Delta = b^1_\Delta \gamma + \mathrm{O}(\gamma^3), \quad c'_\Delta = c^2_\Delta \gamma^2 + \mathrm{O}(\gamma^4), \quad d'_\Delta = d^1_\Delta \gamma + d^3_\Delta \gamma^3 + \mathrm{O}(\gamma^5),$$

in which b^1_Δ, c^2_Δ, and d^j_Δ depend upon a and c and are even in a.

The polynomial $\Delta'(\gamma, a, c)[\nu]$ has three real roots if the discriminant

$$\delta(\gamma, a, c) = 18 b'_\Delta c'_\Delta d'_\Delta + b'^2_\Delta c'^2_\Delta - 4 b'^3_\Delta d'_\Delta - 4 c'^3_\Delta - 27 d'^2_\Delta,$$

is positive. Notice that $\delta(\gamma, a, c)$ is analytic with respect to (γ, a, c) and even with respect to both γ and a.

We prove in Appendix A that $d^1_\Delta = 0$, so that $\delta(\gamma, a, c) = \mathrm{O}(\gamma^6)$. From (iii) we find

$$\delta(\gamma, 0, c) = 3136\gamma^8 + \mathrm{O}(|c|\gamma^8 + \gamma^{10}).$$

Next, since $L_{\gamma,a,0} = L^0_{\gamma,0,0}$, we have $\delta(\gamma, a, 0) = \delta(\gamma, 0, 0)$, so that in the Taylor expansion of δ with respect to γ, a, c there is no monomial of the form $\gamma^n a^m$ with $m \geq 1$. Finally, we take into account the parity properties of δ, and obtain the expansion

$$\begin{aligned}\delta(\gamma, a, c) = \;& \alpha_{216}\, a^2 c \gamma^6 + \alpha_{226}\, a^2 c^2 \gamma^6 + \alpha_{416}\, a^4 c \gamma^6 + \alpha_{426}\, a^4 c^2 \gamma^6 + 3136\gamma^8 \\ & + \mathrm{O}(a^2|c|^3\gamma^6 + a^4|c|^3\gamma^6 + a^6|c|\gamma^6 + |c|\gamma^8 + \gamma^{10}).\end{aligned} \tag{6.9}$$

We calculate the coefficients α_{mp6} in (6.9) in the following way.

1. We first compute explicitly, the coefficients $\widetilde{k}_{2j}(c)$ and $\widetilde{p}_{n,m}(c)$ of the power series (2.4) and (2.5) for $j \leq 2$ and $n+m \leq 4$, and obtain the expansion for the frequency k and the periodic orbit $p_{a,c}$ up to order a^4. This is achieved by induction when identifying the powers of a in (2.7) in which k and $p_{a,c}$ have been replaced by their power series (2.4) and (2.5).

2. Next, using these explicit formulae for k and $p_{a,c}$ up to order a^4, we deduce explicit formulae, here again up to order a^4, for $L^0_{\gamma,a,c}$, $L^1_{\gamma,a,c}$, $R^0_{\gamma,a,c}$, and finally for $D(\gamma, a, c)$ and $\delta(\gamma, a, c)$. The expression obtained for $\delta(\gamma, a, c)$ is exact with respect to γ and c, and up to order 4 in a.

3. Expanding this last formula with respect to γ and c we finally compute α_{mp6} explicitly for $m \leq 4$ and $p \leq 2$.

These long computations have been performed with the help of MAPLE. We finally get

$$\alpha_{216} = 0, \quad \alpha_{226} = \frac{308}{3}, \quad \alpha_{416} = 0, \quad \alpha_{426} = 0,$$

so that

$$\delta(\gamma, a, c) = \frac{308}{3} a^2 c^2 \gamma^6 + 3136\gamma^8 + \mathrm{O}(a^2|c|^3\gamma^6 + a^4|c|^3\gamma^6 + a^6|c|\gamma^6 + |c|\gamma^8 + \gamma^{10}).$$

This quantity is positive if $|c|$ is sufficiently small and $a = \mathrm{o}(|c|^{1/4})$, which we assumed in the last part of the lemma. ■

Remark 6.10 *Instabilities (corresponding to a negative discriminant $\delta(\gamma, a, c)$) occur for $1 \gg a \gg |c|^{\frac{1}{4}}$ if one of the coefficients α_{m16} of $a^m c\gamma^6$ in the Taylor expansion of $\delta(\gamma, a, c)$ does not vanish. The explicit computation of these coefficients for $m \geq 6$ turns out to be completely intractable and we found no theoretical evidence that they should all vanish.*

Appendix A

We show that the coefficient d^1_Δ in the proof of Lemma 6.7 vanishes. The coefficient d'_Δ in (6.7) is obtained from

$$\det A_{\gamma,a,c}[0] = D(\gamma, a, c)[0] = (\mathrm{i}k)^3 a_\Delta \Delta'(\gamma, a, c)[0] = (\mathrm{i}k)^3 a_\Delta d'_\Delta,$$

so that we have to show that the coefficient of the $\mathrm{O}(\gamma)$ term in the expansion of $\det A_{\gamma,a,c}[0]$ vanishes.

We write

$$\begin{aligned} A_{\gamma,a,c}[0] &= -\Pi^0 L_{\gamma,a,c}\Pi_{\gamma,a,c}\Pi^0 = -\Pi^0(L^0_{\gamma,a,c} + L^1_{\gamma,a,c})(\Pi^0 + \Pi^1_{\gamma,a,c})\Pi^0 \\ &= -L^0_{\gamma,a,c}\Pi^0 - \Pi^0 L^1_{\gamma,a,c}(\Pi^0 + \Pi^1_{\gamma,a,c})\Pi^0 - L^0_{\gamma,a,c}\Pi^0\Pi^1_{\gamma,a,c}\Pi^0, \end{aligned}$$

in which $\Pi^1_{\gamma,a,c}$ denotes the sum in the right hand side of (6.3),

$$\Pi^1_{\gamma,a,c} = \frac{1}{2\pi\mathrm{i}} \int_{C(0,1)} \sum_{n \geq 1} R^0_{\gamma,a,c}(s) \left(L^1_{\gamma,a,c} R^0_{\gamma,a,c}(s)\right)^n \mathrm{d}s.$$

Recall that $L^1_{\gamma,a,c} = k(\partial_x + \mathrm{i}\gamma)(p_{a,c}\cdot)$. Then, we can write

$$A_{\gamma,a,c}[0] = -L^0_{\gamma,a,c}\Pi^0 - (\partial_x + \mathrm{i}\gamma)\Pi^0 B \Pi^0, \qquad \text{(A.1)}$$

in which B is a bounded linear operator. Here, we have used the fact that each term in the sum in the explicit formula for $\Pi^1_{\gamma,a,c}$ contains at least one factor $L^1_{\gamma,a,c}$, and that $(\partial_x + \mathrm{i}\gamma)$ commutes with Π^0 and $R^0_{\gamma,a,c}(s)$. From (A.1), we obtain

$$A_{\gamma,a,c}[0] = \mathrm{i}k \begin{pmatrix} \gamma\eta_{0,0} & \gamma\eta_{0,1} & \gamma\eta_{0,-1} \\ (\gamma+1)\eta_{1,0} & (\gamma+1)\eta_{1,1} & (\gamma+1)\eta_{1,-1} \\ (\gamma-1)\eta_{-1,0} & (\gamma-1)\eta_{-1,1} & (\gamma-1)\eta_{1,-1} \end{pmatrix}, \tag{A.2}$$

in which the coefficients $\eta_{i,j}$ depend analytically upon γ, a, and c. Then $\det A_{\gamma,a,c}[0] = (\mathrm{i}k)^3\gamma(\gamma^2-1)\det(\eta_{i,j})$, so that the $\mathrm{O}(\gamma)$ term in $\det A_{\gamma,a,c}[0]$ vanishes if the determinant $\det(\eta_{i,j})$ vanishes at $\gamma = 0$. From the relation $A_{-\gamma,a,c}[0]S = -SA_{\gamma,a,c}[0]$ and the explicit form (6.6) of S, a direct calculation implies that at $\gamma = 0$ the matrix $(\eta_{i,j})$ has the form

$$(\eta_{i,j})_{|\gamma=0} = \begin{pmatrix} \eta^0_{0,0} & \eta^0_{0,1} & \eta^0_{0,1} \\ \eta^0_{1,0} & \eta^0_{1,1} & \eta^0_{1,-1} \\ \eta^0_{1,0} & \eta^0_{1,-1} & \eta^0_{1,1} \end{pmatrix},$$

and a sufficient condition for its determinant to vanish is that $\eta^0_{1,1} = \eta^0_{1,-1}$. We compute these two coefficients with the help of $A_{0,a,c}[\lambda]$, and show that they are equal, so that $\det(\eta_{i,j}) = 0$. By arguing as for (A.2) we find

$$A_{0,a,c}[\mathrm{i}k\nu] = \mathrm{i}k \begin{pmatrix} \nu & 0 & 0 \\ \nu\delta^0_{1,0} + \eta^0_{1,0} & \nu\delta^0_{1,1} + \eta^0_{1,1} & \nu\delta^0_{1,-1} + \eta^0_{1,-1} \\ \nu\delta^0_{-1,0} - \eta^0_{1,0} & \nu\delta^0_{-1,1} - \eta^0_{1,-1} & \nu\delta^0_{-1,-1} - \eta^0_{1,1} \end{pmatrix},$$

in which $\delta_{i,j}$ are analytic functions in a and c, and $\eta^0_{i,j}$ are precisely the coefficients before. Since $\det A_{0,a,c}[\lambda] = \alpha\lambda^3$, the coefficient of the linear term in ν in $\det A_{0,a,c}[\mathrm{i}k\nu]$ vanishes, so that we have $(\eta^0_{1,1})^2 = (\eta^0_{1,-1})^2$. The Taylor expansion in (a, c) of $\eta^0_{1,1} + \eta^0_{1,-1}$ can be computed from the Taylor expansion of the periodic wave $p_{a,c}$ and the explicit formulae (6.5), (6.3), and (6.1). At second order in a, we find

$$\eta^0_{1,1} + \eta^0_{1,-1} = \frac{1}{48}a^2c^2 + \mathrm{O}(a^3c + a^2c^3),$$

so that $\eta^0_{1,1} + \eta^0_{1,-1}$ does not vanish in some open set of sufficiently small values of a and c. Then $\eta^0_{1,1} - \eta^0_{1,-1} = 0$ in this open set, and we conclude that $\eta^0_{1,1} - \eta^0_{1,-1} = 0$ due to analyticity.

Appendix B

Lemma B.1 *Let X be a Banach space and let Π and Π' be two projections in $\mathcal{L}(X)$. Denote by E and E' the range of Π and Π', respectively.*

If Π *has finite rank and*

$$\|\Pi - \Pi'\|_{\mathcal{L}(X)} < \min\left(\frac{1}{\|\Pi\|_{\mathcal{L}(X)}}, \frac{1}{\|\Pi'\|_{\mathcal{L}(X)}}\right),$$

then the two projections Π *and* Π' *realize isomorphisms between* $E' \to E$ *and* $E \to E'$*, respectively. In particular, they have the same finite rank.*

Proof. Observe that $\Pi \circ \Pi' \circ \Pi \in \mathcal{L}(E)$ and that

$$\Pi \circ \Pi' \circ \Pi = \mathrm{id}_E + \Pi \circ (\Pi' - \Pi) \circ \Pi.$$

Then, since $\Pi|_E = \mathrm{id}_E$ we get that

$$\|\Pi \circ (\Pi' - \Pi) \circ \Pi\|_{\mathcal{L}(E)} < \|\Pi\|_{\mathcal{L}(X)} \; \|(\Pi' - \Pi)\|_{\mathcal{L}(X)} < 1.$$

Thus, $\Pi \circ \Pi' \circ \Pi$ is one to one from E onto E. In particular,

$$E = \Pi \circ \Pi' \circ \Pi(E) = \Pi \circ \Pi'(E) \subset \Pi(E') \subset E,$$

so that

$$\Pi(E') = E, \qquad \dim E \leq \dim E' \leq \infty.$$

Similarly we prove

$$\Pi'(E) = E', \qquad \dim E' \leq \dim E < \infty.$$

This ensures that Π and Π' have the same finite rank and realize isomorphisms between $E' \to E$ and $E \to E'$, respectively.

■

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