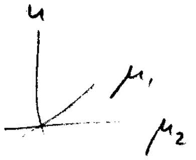
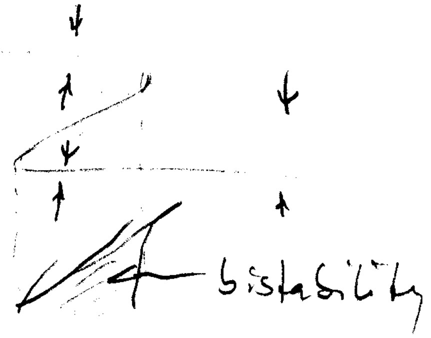
RDS $u_t = D\Delta u + f(u) \in \mathbb{R}^N$

$f \to$ cusp

- $f(0)=0$, $0 \in \operatorname{spec} f'(0)$ alg. simple
  e, e^* ev, adj. ev
- $e^*, \circ f''(0)[e,e] = 0$

↳ typically $\dot u = f(u)$





PDE? interfaces?
coherent structures?

ODE Methods $u = u(x-ct)$ TW ODE - 'trick'

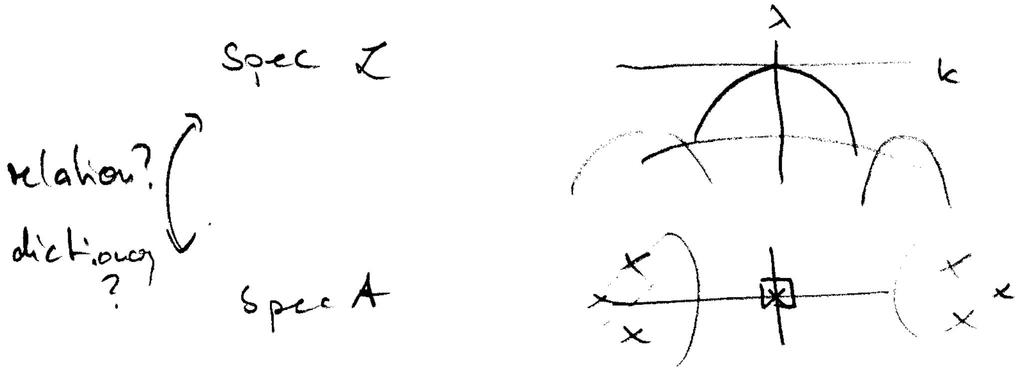
$$Du_{\xi\xi} + cu_\xi + f(u) = 0$$

$$\mathbb{R}^{2N} \ni \begin{cases} u_\xi = v \\ v_\xi = -D^{-1}(cv + f(u)) \end{cases}$$

PDE $\leftrightarrow$ ODE relation: linearization

PDE $\quad \mathcal{L}u = Du_{xx} + f'(0)\,u \quad \in L^2(\mathbb{R})$

ODE $\quad \begin{pmatrix}u\\v\end{pmatrix}_\xi = \underbrace{\begin{pmatrix}0 & 1\\ -Df' & 0\end{pmatrix}}_{A}\begin{pmatrix}u\\v\end{pmatrix} \quad \in \mathbb{R}^{2N}$



<u>Lemma</u> $\nu \in \operatorname{spec} A$ alg. mult. p $\iff$ $\lambda \in \operatorname{spec} \mathcal{L}$, $\lambda(k) \sim k^p$

<u>Pf</u> disp. relation $e^{\lambda t + \nu x} u_0 \sim$

$$d(\lambda,\nu) = \det\left(D\nu^2 + f'(0) - \lambda\right) = 0$$

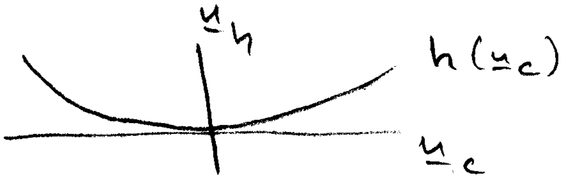
$\lambda \in \operatorname{spec} \mathcal{L} \iff d(\lambda, ik) = 0$ for some $k \in \mathbb{R}$

$\nu \in \operatorname{spec} A \iff d(0,\nu) = 0$

Here: $\lambda, \nu \sim 0$: $\quad d \sim \lambda + ..\nu^p + \dots \qquad \bowtie$

<u>Reduction</u> : center-manifolds (rigorous, ptwise)

TWODE $\underline{u}_\xi = A\underline{u} + f(\underline{u})$ $\underline{u} = \underline{u}_c + \underline{u}_h$



$$\underbrace{\underline{u}_c' = A\underline{u}_c}_{\text{alg. growth}} \qquad \underbrace{\underline{u}_h' = A\underline{u}_h}_{\text{exp. decay/growth}}$$

$\hookrightarrow$ $\underline{u}_c' = A\underline{u}_c + f(\underline{u}_c + h(\underline{u}_c)) \to$ all small bdd sol'

$A = \begin{pmatrix} 0 & 1 \\ -D^{-1}f' & 0 \end{pmatrix}$: $\underline{u}_c = A \cdot \begin{pmatrix} e \\ 0 \end{pmatrix} + B \begin{pmatrix} 0 \\ e \end{pmatrix}$ [Vanderbauwhede]

$$\begin{cases} A_x = B \\ B_x = \hat{\mu}_1 + \hat{\mu}_2 A + A^3 + \hat{c}B + \dots \end{cases}$$

Compute $\hat{\mu}_1, \dots$ need adjoint $\begin{pmatrix} 0 & -f'^T D^{-1} \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 \\ De^* \end{pmatrix} \to$ ker $\begin{pmatrix} De^* \\ 0 \end{pmatrix} \to$ princ. ev.

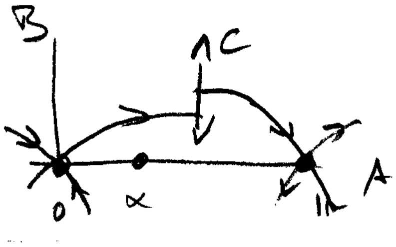
$-\hat{c} \div \begin{pmatrix} 0 \\ De^* \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -D^{-1}ce \end{pmatrix}$ & normalization $\langle De^*, e \rangle > 0$

Scaling in bistable regime:

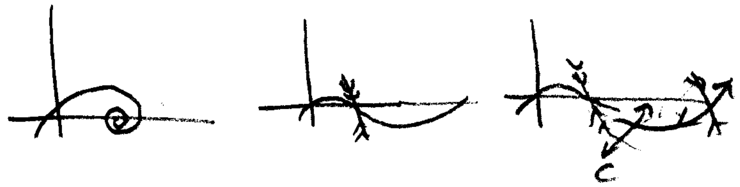
$A_x = B$

$B_x = \cdot \; cB - A(1-A)(A-\alpha) + \dots$ $0<\alpha<1$

What's the point? Phase plane methods,
Poincaré-Bendixson,
shooting
<u>to any order</u>



vary c



dict.

PDE stable ⟷ ODE hyperbolic

absolute inst. ⟺ spiral

conv. inst ⟺ node

$\lambda = 0$ alg simple for $\mathcal{L}$ $\overset{}{\underset{(*)}{\Longleftrightarrow}}$ transverse crossing

u_* front, $\mathcal{L}u = Du_{\xi\xi} + cu_\xi + f'(u_*)u$

(*) Pf: ODE: Melnikov $\neq 0$ ~ transverse crossing ✓

- Solve $0 = u_{xx} + cu_x + f(u)$ near u_* robustly (or equiv 1st order ODE)

$\partial_c|_{u=u_*}$: $\mathcal{L}(\partial_c u) + \partial_x u_* = 0$, unique sol if $\partial_x u_* \notin \mathrm{Rg}\,\mathcal{L}$

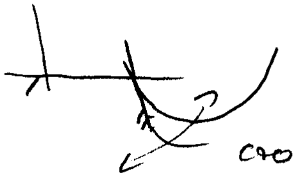
$\rightarrow$ Melnikov $\langle \partial_x u_*, \psi \rangle \neq 0$

PDE: $\lambda u = u_{xx} + cu_x + f'(u_*)u$ near $u = \partial_x u_*$, ev $\lambda = 0$

$\partial_\lambda|_{u=u_*'}$: $\partial_x u_*^{\psi} = \mathcal{L}u$, unique sol if $\partial_x u_* \notin \mathrm{Rg}\,\mathcal{L}$

M

How interprete st-unst fronts:



$u_*(\xi) \to u_\pm$

...

Fredholm $Rg\,(\mathcal{L}-\lambda)$ closed, dim ker, Rg $< \infty$

index $i(\mathcal{L}-\lambda) = \dim \ker - \dim Rg$

"# of free parameters" in solving $(\mathcal{L}-\lambda)u = g$

Ess Spec : Σ_{ess} not Fredholm index 0

Pt Spec Fredholm index 0, not inv'ble

Morse Index $i_\pm = \{ \# \nu \mid \mathrm{Re}\,\nu > 0,\ \overset{\det(D\nu^2 + c\nu + f'(u_\pm) - \lambda)}{d_\pm(\lambda,\nu)} = 0 \}$

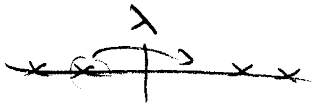
(Dict.) **Thm** $\underset{\text{PDE}}{i} = \underset{\text{ODE}}{i_- - i_+}$ (for all λ where defined)

'Pf'

Re $\nu > 0$ in ODE $\left\{ \begin{array}{l} \text{decay } \xi \to -\infty \to i_- = \#\{\text{shooting parameters}\} \\ \text{growth } \xi \to +\infty \to i_+ = \#\{\text{conditions}\} \end{array} \right.$

How to compute $i_\pm$? Homotopy in λ !

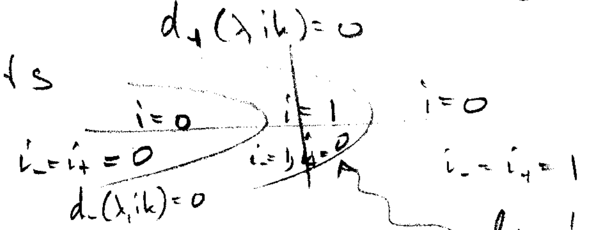
$\lambda \to +\infty$: inv'ble, index 0 $\to$ well-posedness



$$\frac{d\nu}{d\lambda} = \frac{1}{\frac{d\lambda}{d\nu}} = -\frac{1}{c_g}$$

$e^{\lambda t + \nu x} = e^{i\omega t - ikx} \rightsquigarrow \frac{d\lambda}{d\nu} = -\frac{d\omega}{dk} = -c_g$

fronts



$c_g = -c_f < 0$

disp' curves ~ Fredholm borders

Exp weights:

$L^2_{\eta_-, \eta_+}$, $|u|^2 = \int_0^\infty |u e^{\eta_+ x}|^2 + \int_{-\infty}^0 |u e^{\eta_- x}|^2$

isom. to L^2 $\quad v = u e^{\eta_\pm x}, \ x \gtrless 0$

$$\tilde d_+(\lambda, \nu) = d_+(\lambda, \nu - \eta_+)$$
$$\tilde d_-(\lambda, \nu) = d_-(\lambda, \nu - \eta_-)$$

$i^{\eta_\pm} = \tilde i_- - \tilde i_+$, computed w.r.to $\tilde d$

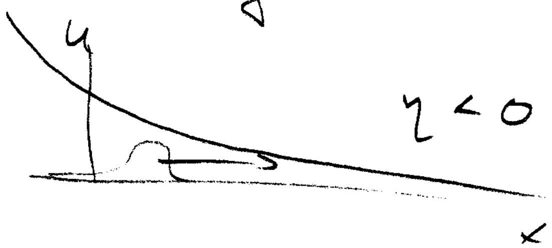
Moving spectra around

e.g. $\lambda|_{\eta=0} = \lambda_0 - c_g ik + O(k^2)$

$$\lambda^{\eta_\pm} = \lambda_0 - c_g(ik - \eta) + —$$
$$= \lambda_0 + c_g \eta - c_g ik + —$$

$$\boxed{\frac{d\lambda}{d\eta} = c_g}$$

$u_t = -c_g u_x$



stabilization

Profile $u_*(\xi) \to u_\pm$

Disp rel: $e^{\lambda t + \nu x}$

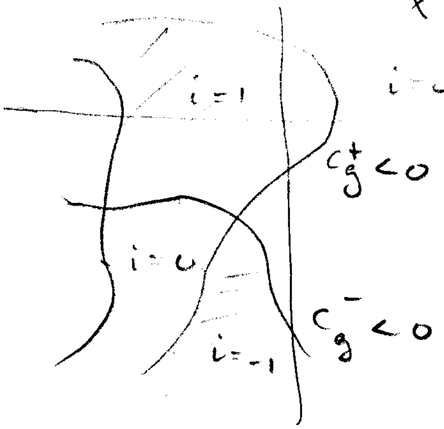
$$d_\pm(\lambda,\nu) = \det\left(D\nu^2 + c\nu + f'(u_\pm)\right) = 0$$

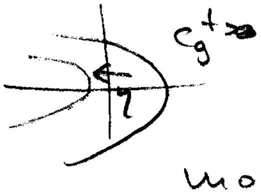
MI $\quad i_\pm(\lambda) = \#\{\mathrm{Re}\,\nu > 0\}$, ~~dl~~

FI $\quad i = i_- - i_+$

Fredholm boundary $\quad d_\pm(\lambda, ik) = 0$

$\lambda \in \mathbb{C}$ $\quad L^2_{\eta_\pm}$ $\quad \frac{d\lambda}{d\eta} = c_g$





moving Fredholm boundaries

How far can we go?

How far can we go

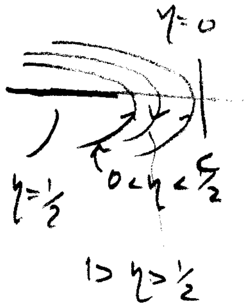
$\lambda u = u_{xx} + c u_x$

$d = \nu^2 + c\nu - \lambda$

$= -k^2 + 2ik\eta + \eta^2 + cik - c\eta - \lambda$

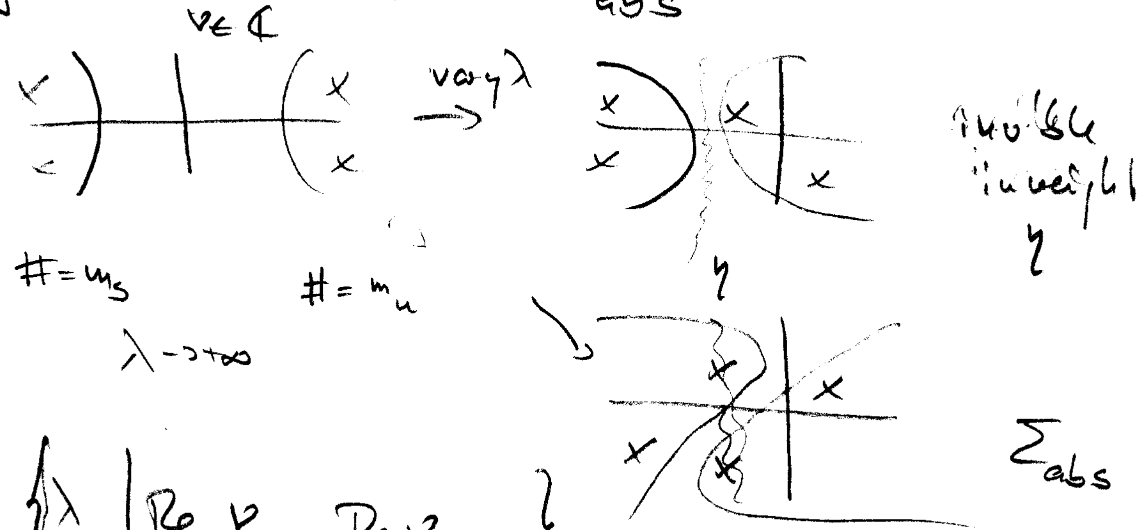
$\Rightarrow \lambda = (+\eta^2 - c\eta) + ik(c - 2\eta) - k^2$

$\geq -\frac{c^2}{4}$



best weight → abs spec Σ_{abs}

General

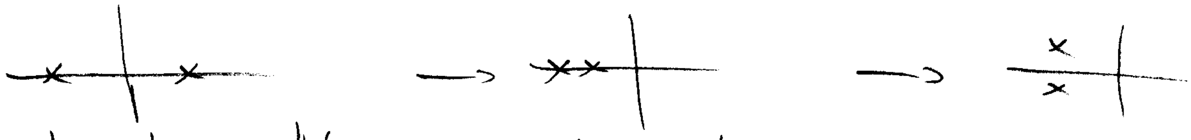


$\Sigma_{abs} = \{\lambda \mid Re\, \nu_{m_s} = Re\, \nu_{m_s+1}\}$

$Re\, \nu_1 \leq Re\, \nu_2 \leq \dots$

Example

$\lambda = 0 \to$ ODE linearization



hyp' → ins'ble

node → ins'ble in weight → convectively unstable

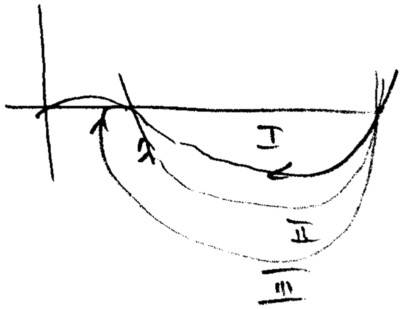
focus → ~~ins'ble~~ $0 \in \Sigma_{abs}$ → abs unst.

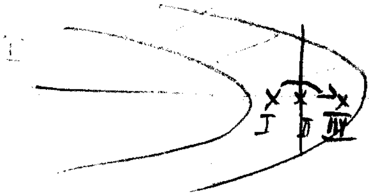
$\lambda \notin \Sigma_{abs} \Rightarrow$ ex. $\eta_\pm$: $\mathcal{L}-\lambda$ index 0, with Morse index $m_u^\pm$

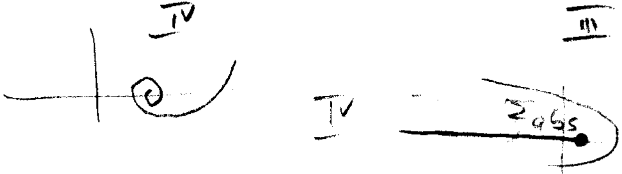
$$\Sigma_{expt} := \{\lambda \mid \text{ev of } \mathcal{L} \text{ in } L^2_{\eta_\pm}, \lambda \notin \Sigma_{abs}\}$$

(independent of $\eta_\pm$ as long as m_u unchanged, analytic ext' of Evans fct)

Ex







transv' crossing $\Longleftrightarrow$ λ alg simple in Σ_{expt}

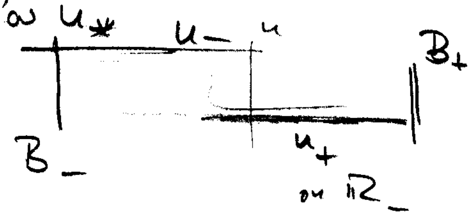
In the real world $x \in [-L, L]$, $B_\pm u(\pm L) = 0$

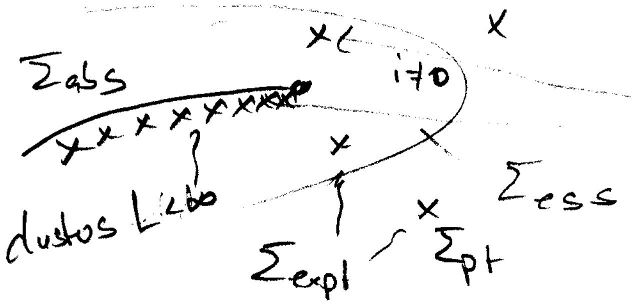
(Rk $B_\pm = \begin{cases} m_u \\ 2N - m_u \end{cases}$)

Thm $\mathrm{spec}_L \mathcal{L} \xrightarrow[\substack{\text{mult'} \\ L\to\infty}]{\text{set \&}} \Sigma_{abs} \cup \Sigma_{expt} \cup \Sigma_{bdy}$

$$\Sigma_{bdy} := \bigcup_{+,-} \Sigma_{expt}\, \underset{\substack{| \\ \text{const. coeff}}}{\mathcal{L}_\pm} \Big|_{\mathbb{R}_\mp}$$

subst' $u_\pm$ for u_*





exp growing for $i<0$

str. localized for $i>0$

Similar for $u_* \to u_{per}$

$\nu \to$ Floq exp'

- time-periodic

PDE $u_t = Du_{xx} + f(u) + cu_x$

"Mod ODE"

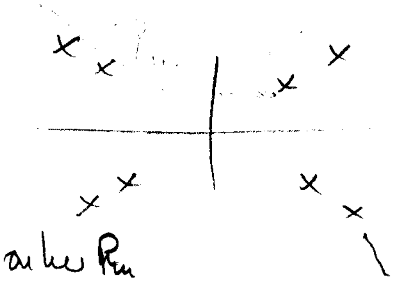
$$u_x = v$$
$$v_x = D^{-1}(\omega u_\tau - cv - \overset{P_m}{)} f(u))$$

$(u,v) \in H^1_{per} \times H^{1/2}_{per}(0,2\pi)$ $=: Y$

Fourier

$H^m \sim \sum (|k|^m |u_k|)^2 < \infty$

$$P_m : u \longmapsto \sum_{|k| \le m} u_k e^{ikt}$$



on ker P_m

$\nu_k = \pm\sqrt{d\, ik}$

- MODE trivial on ker P_m
- ODE on Rg P_m
- continuous $P_m \xrightarrow[m\to\infty]{} id$

→ hyperbolic ~ invertible | W^u, W^s ∞-dim. but still W^c finite dim and

<u>Thm</u> $i = i_- - i_+$

$= i^m_- - i^m_+$ (on Rg P_m, m large)

$$i = i(\mathcal{L}) = i(\phi)$$

$\mathcal{L}u = \omega u_\tau - Du_{xx} - cu_x - f'(u_*)u$ on $L^2(S^1 \times \mathbb{R})$

$\phi(u_0) = u(\tau = 2\pi) - u(\tau = 0)$ on $L^2(\mathbb{R})$

Example Homogeneous oscillations
+ inhomogeneity

$u_t = Du_{xx} + f(u) + \epsilon g(x)$, $u = u_*(\omega t)$
periodic sol
at $\epsilon = 0$

$u_x = v$

$v_x = D^{-1}(\omega u_\tau - f(u) - \epsilon g(x))$

$\underline{\epsilon = 0}$ circle of eq' $\begin{pmatrix} u_*(\cdot + \theta) \\ 0 \end{pmatrix}$ ~

Dict

PDE	ODE
autonomous	symmetry
symmetry	nonautonom
gradient	Hamiltonian
reflection	reversibility

lin. $u_x = v$

$v_x = D^{-1}(\omega u_\tau - f'(u_*)u)$

evP: $D\nu^2 u + f'(u_*)u - \omega u_\tau = 0$, 2π-per

↳PDE $\tilde{u}_t = D\tilde{u}_{xx} + f'(u_*)\tilde{u}$

$\tilde{u} = e^{\alpha t} e^{\nu x} u(\omega t)$

↳ $\alpha u + \omega u_\tau = D\nu^2 + f'u$, $\underline{\alpha \text{ Floq' exp.}}$

~~α~~ $\nu = ik$, $\alpha = \alpha(k)$, $\alpha = 0$ for $k = 0$,
with $u = u_*'$, simple exp'

$\alpha(k) = -d_{eff}k^2 + O(k^4)$
} disp' relation

$d_{eff} \neq 0 \xrightarrow{\text{dict.}}$ $\nu = 0$ alg. double

Center manifold (... some subtleties...)

ev $e_0 = \begin{pmatrix} u_*' \\ 0 \end{pmatrix}$, $e_1 = \begin{pmatrix} 0 \\ u_*' \end{pmatrix}$

L^2-adj; $e_0^* = \begin{pmatrix} 0 \\ D\underline{u^*_{\cancel{*}}} \end{pmatrix}$, $e_1^* = \begin{pmatrix} Du^* \\ 0 \end{pmatrix}$

$\omega u_\tau^* = (f')^T(u_*)\, u^*$

$\to$ 2d CMfd $\sim \underbrace{S^1}_{\theta} \times \underbrace{\mathbb{R}}_{k}$

$u = u_*(\cdot + \theta(x)) + k e_1(\cdot + \theta(x))$
$+ \dots$

$\theta_x = k + \cancel{}$ $k \to k_{new}$

$k_x = g(k) + \overset{-\omega}{\cancel{O(\omega)}} + \dots$ (no θ-dep. because of symmetry)

ω-dependence: $\left(\begin{pmatrix} 0 \\ Du^* \end{pmatrix}, \begin{pmatrix} 0 \\ D^{-1}\partial_\tau u_{*\cancel{*}} \end{pmatrix} \right)_{L^2(S^1)} > 0$

$\hookrightarrow$

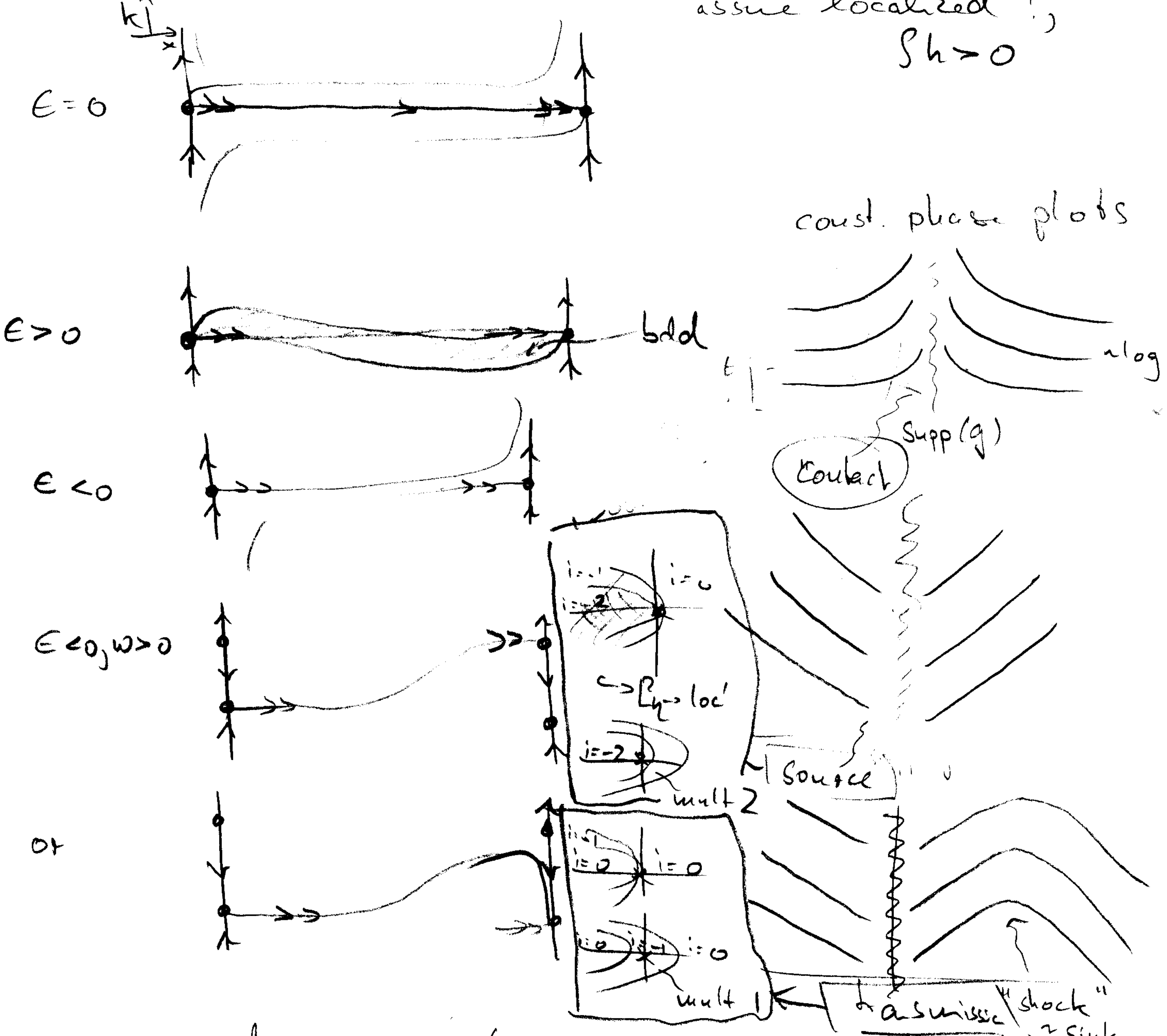
$g = ?$ $k_x = 0$ for $\omega_0 = g(k_0)$

$\Rightarrow$ sol. $u \sim u_*(k_0 x - \omega_0 t) \to$ wave train

"Nonlinear disp. rel."

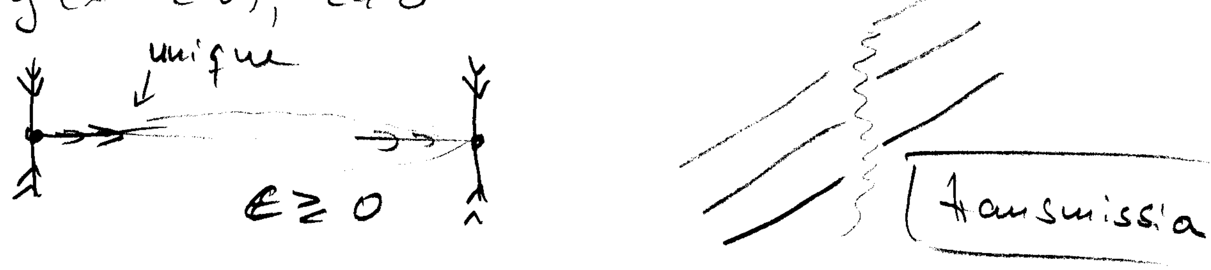
$\omega = g_2 k^2 + \dots$

$k_x = -\omega + k^2 \quad - \epsilon h(x)$ ← from 'inhom' assume localized !, $\int h > 0$



$c_g = \frac{d\omega}{dk} \sim k$ (as in Melnikov ⇒ transverse crossing)

If $g = g(x-ct)$, $c \neq 0$



Radial sol's, higher dimensional

$$D\left(u_{rr} + \frac{n-1}{r} u_r\right) + f(u) = 0 \qquad (= \omega u_t)$$

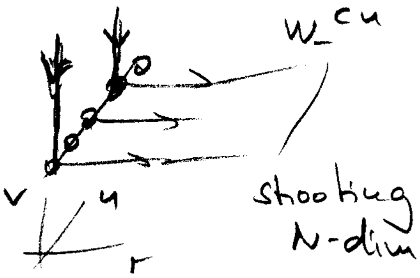
$$u_r = v$$
$$v_r = -\frac{n-1}{r} - D^{-1} f$$

two regimes, then matching:

- $r \sim 0 \quad \rightarrow \quad \tau = \log r$

$$u_\tau = r v$$
$$v_\tau = -(n-1) v - r D^{-1} f$$
$$r_\tau = r$$

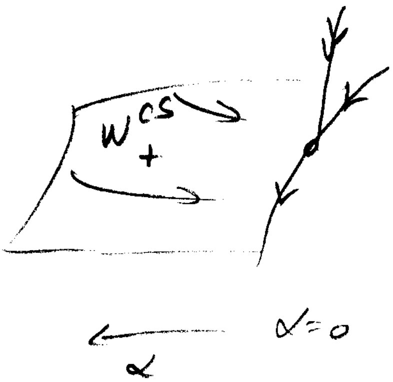
eq' $u = *$, $v = r = 0$



- $r \to \infty \qquad \alpha = 1/r$

$$u_r = v$$
$$v_r = -\alpha(n-1) v - D^{-1} f$$
$$\alpha_r = -\alpha^2$$

$\alpha = 0 \rightsquigarrow$ 1d problem, e.g. Cmfd



the $W_+^{cs} \cap W_-^{cu}$ ...

Dict reduced αv-term

$\hookrightarrow$ effective "transverse" stability

Fredholm $= -i_\infty$, Morse at $+\infty$

Spiral instabilities $\quad u(r, \omega t - \varphi)$

φ used to be called τ!

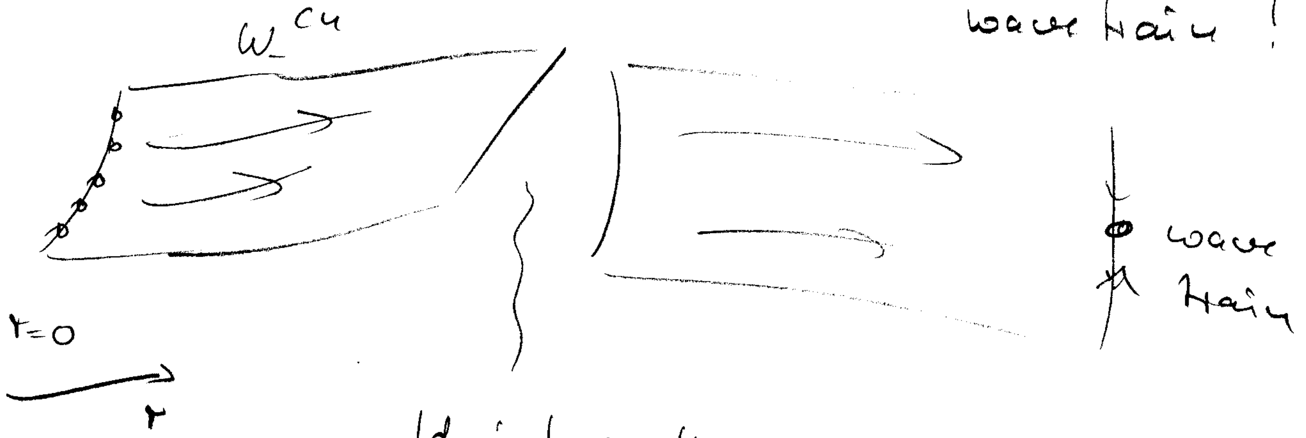
(P_m)

$u_r = v$

$v_r = -\frac{1}{r}v - \frac{1}{r^2}u_{\varphi\varphi} - D^{-1}\left(f(u) - \omega u_\varphi\right)$

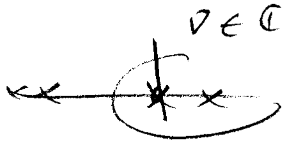
$r=\infty$: $\quad Du_{rr} + f(u) = \omega u_\varphi \rightsquigarrow u(\overset{2\pi\text{-per}}{\overbrace{kr-\varphi}})$

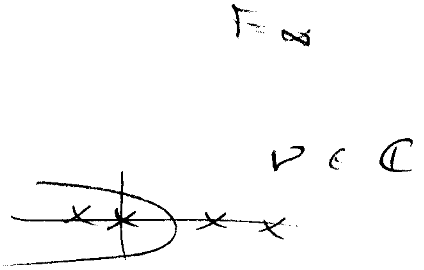
wave train!



1d-intersection

$\partial_\varphi u_*$

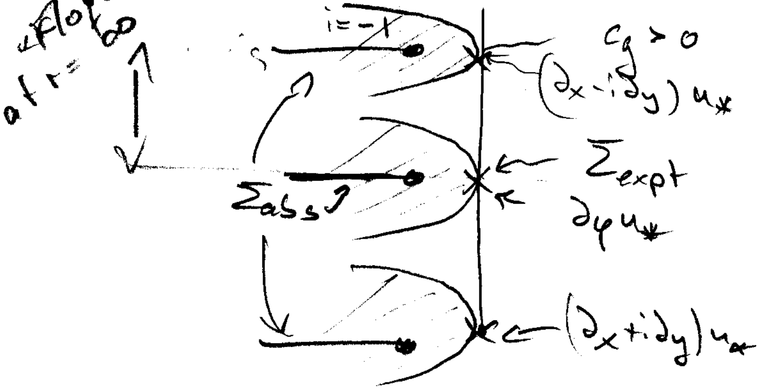


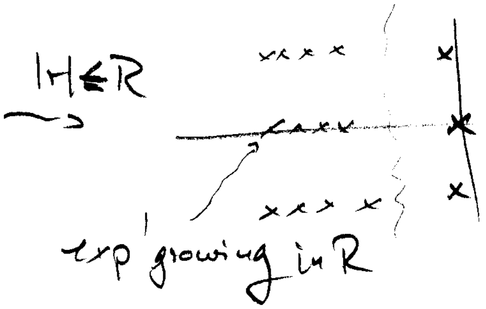


codim-one,

transverse in ω

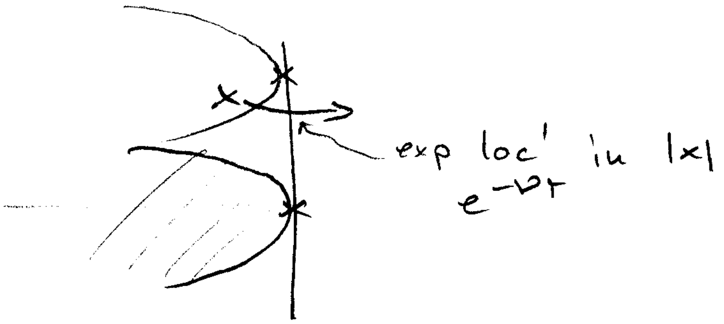
$$\mathcal{L}u = D\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}\right) - \omega u_\varphi + f'(\underset{}{\overset{\text{spiral}}{u_*}})u$$





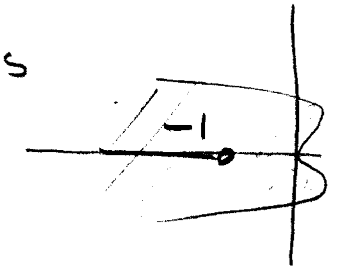
List of instabilities

pt



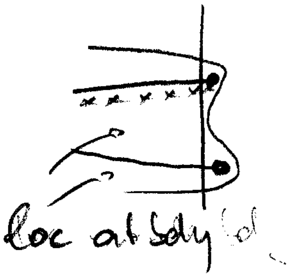
$\nu \sim (\omega_H - \omega_\varphi)^2$ } meandering

ess



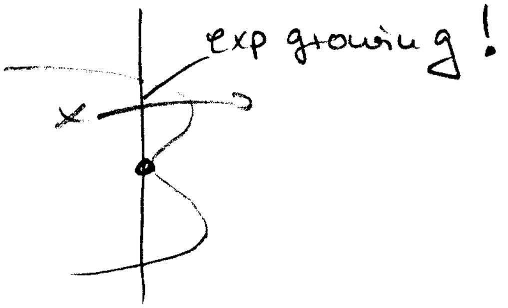
transient, convective

abs



absolute, also in bdd domains

expt pole



breakup & meander

⋮

- additional Σ_{ess} → $i=+1$ possible, Σ_{abs} localized at core
- resonances sometimes forced because of

Floquet periodicity → robust doubling