The impossible period-doubling of a spiral wave

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joint work with
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Simple and complicated patterns

Patterns in the Belousov-Zhabotinsky reaction

[Park, Lee], [Zhou, Ouyang], [Agladze, Krinsky, Pertsov]
From simple to complicated dynamics

Routes to chaos in dynamical systems

\[ \frac{d}{dt} u = f(u; \mu), \quad u \in X = \mathbb{R}^N, \mu \in \mathbb{R} \]

Hopf bifurcations

1 frequency \( \rightarrow \) 2 frequencies \( \rightarrow \) 3 frequencies
periodic orbit \( \rightarrow \) two-torus \( \rightarrow \) strange attractors

Period doubling cascades

Occurrence of Strange Axiom A Attractors Near Quasi Periodic Flows on \( T^m, m \geq 3 \)

S. Newhouse\(^1\), D. Ruelle\(^3\), and F. Takens\(^3\)*
From simple to complicated patterns — spiral waves

\[
\frac{du}{dt} = D \Delta u + f(u), \quad u \in X = C^2(\mathbb{R}^n, \mathbb{R}^N)
\]

Spirals ↔ periodic orbits

Two generic instability mechanisms (ex. FHN, Roessler):

- Hopf bifurcation
  - two frequencies

- Period-doubling
  - half frequency

— click on images to play movies —
Bifurcations and spiral waves — oddities

More Hopf instabilities...

Drift

Breakup
Euclidean symmetry

\[ \gamma \in SE(2), \text{ rotations and translations} \]
\[ \gamma = (\varphi, z) \in S^1 \times \mathbb{C} \quad \gamma \cdot x = e^{i\varphi}(x + z) \in \mathbb{R}^2 \sim \mathbb{C} \]

\[ u(t, x) \text{ solution} \iff u(t, \gamma^{-1}x) \text{ solution} \]

Spirals are relative equilibria

\[ u_{sp}(t, x) = u_{sp}(0, \gamma^{-1}(t)x) \quad \gamma(t) = \exp(\omega_{sp}t \partial \varphi) \]
Bifurcations from Relative Equilibria

Reduction to principal fiber bundle

$\gamma \in SE(2) \times V$

\[ \dot{\gamma} = \gamma a(v) \quad \text{“group”} \]

\[ \dot{v} = h(v) \quad \text{“shape”} \]
Resonances and drift

\[ \dot{\phi} = \omega_{sp} \quad \text{rotation} \]
\[ \dot{z} = e^{i\phi} v \quad \text{translation} \]
\[ \dot{v} = h(v) \quad \text{Hopf} \]

**Periodic Orbit**
\[ v(t) = \sum_k v_k e^{-ik\omega_H t} \]

**Position**
\[ \dot{z} = \sum_k v_k e^{i(\omega_{sp} - k\omega_H) t} \]

Unbounded motion if \( \omega_H = \omega_{sp}/k \) for some \( k \in \mathbb{Z} \)
The paradox

Spirals are relative *equilibria* $\implies$ period-doubling is non-generic

... yet is *is* observed

More precisely...

Linearizing at a spiral wave $u_{sp}(t, x)$ we find

**Linearized period map:** $\partial_u \Phi_{2\pi/\omega_{sp}}$, where $u(t) = \Phi_t(u(0))$ is the flow map

**Linearization in corotating frame:** $\mathcal{L} = D\Delta + \omega_{sp}\partial_\varphi + f'(u_{sp})$

Since spirals are equilibria,

$$\partial_u \Phi_{2\pi/\omega_{sp}} = e^{\mathcal{L}(2\pi/\omega_{sp})}$$

The doubling eigenvalue $-1$ cannot be simple!

... but we would expect multiple eigenvalues to split generically.
An explanation with caveats

\[ \lambda = -1 \] is double eigenvalue of \( \partial_u \Phi \) since
\[ \alpha = \pm i \omega_{sp}/2 \] are eigenvalues of \( \mathcal{L}_* \).

Problems:

- **Genericity**: Why is the Hopf frequency in exact resonance?
- **Drift**: If there is an eigenvalue at \( \lambda = i \omega_{sp}/2 \), we expect drift!
Instabilities — linearization

Reaction-diffusion system

$$\partial_t U = D \Delta U + F(U; \mu)$$

Spiral waves as rotating waves

$$U(t, x) = U_{sp}(r, \varphi - \omega_{sp} t)$$

Linearization in corotating frame

$$\mathcal{L}_{sp} U = D \Delta U + F'(U_{sp}; \mu) U + \omega_{sp} \partial_\psi U$$

**Stability:** $\text{Re spec } \mathcal{L} \leq 0$

Eigenvalues enforced by symmetry

- $\lambda = 0$ — rotation
- $\lambda = \pm i \omega$ — translation
Unbounded domains — the essential spectrum

Decompose the spectrum into continuous and discrete part:

\( \text{spec } \mathcal{L}_{sp} : \quad \mathcal{L}_{sp} - \lambda \) not invertible
\( \text{spec}_{ess} \mathcal{L}_{sp} : \quad \mathcal{L}_{sp} - \lambda \) not Fredholm of index 0
\( \text{spec}_{pt} \mathcal{L}_{sp} : \quad \mathcal{L}_{sp} - \lambda \) Fredholm of index 0, not invertible

Localized changes of the spiral shape are compact perturbation of \( \mathcal{L}_{sp} \), and therefore leave \( \text{spec}_{ess} \mathcal{L}_{sp} \) unchanged

\[ \implies \]

Essential spectrum \( \sim \) behavior in the far field
Point spectrum \( \sim \) behavior in the core
Spectra of spiral waves

Spiral waves converge to wave trains

\[ U_{sp}(r, \varphi - \omega_{sp}t) \sim U_{wt}(kr + \varphi - \omega_{sp}t) \text{ for } r \to \infty, \]

they are \textit{asymptotically Archimedean}

**Theorem** [Sandstede & AS]

The essential spectrum of \( \mathcal{L}_{sp} \) is given by the Floquet spectrum of the wave trains.
Spectra of wave trains

Instabilities of wavetrains close to homogeneous period-doubling

Floquet theory: period-doubling of wave trains is robust \( \sim \) spatio-temporal symmetry breaking

\[ \text{Maximum at } i\omega_{sp}/2 \text{ is robust:} \]

\[ \lambda \leftrightarrow \bar{\lambda} \]
\[ \lambda \leftrightarrow \lambda + i\omega_{sp} \]

\[ \text{fix max} \]
In a large disc $|x| \leq R$, with "compatible" boundary conditions:

$$\text{spec} |x| \leq R \mathcal{L}_{sp} \xrightarrow{R \to \infty} \text{spec}_{\text{abs}} \mathcal{L}_{sp} \cup \text{spec}_{\text{expt}} \mathcal{L}_{sp} \cup \text{spec}_{\text{bdy}} \mathcal{L}_{sp}$$

Absolute spectra are determined by wave trains only

Robust "absolute" period-doubling in large domains
A first summary

- Spirals resemble wave trains in the far field.
- Wave trains possess an additional translational symmetry.
- Period-doubling is symmetry-breaking of wave trains.
- Rigorous decomposition on the linearized level:
  
  \[
  \text{spiral core} \longleftrightarrow \text{point spectrum} \quad \longleftrightarrow \quad \text{far field} \longleftrightarrow \text{essential and absolute spectra}
  \]

- In unbounded and large bounded domains, period-doubling is typical when caused by essential or absolute spectrum.

- *Eigenfunctions predict a stationary line defect.*

- *In the Roessler system, the instability appears to be caused by boundary spectrum which happens to be resonant for a similar reason . . .
Drift?

In a fixed bounded domain, the instability caused by the first eigenvalue is a resonant Hopf bifurcation, so we predict drift: We plot the position of the spiral tip and wait...

and wait some more...
Proofs: spatial dynamics \iff functional analysis

\[ \partial_t u = D \Delta u + f(u) \]

\[ 0 = \omega u_{\varphi} + D(u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\varphi\varphi}) + f(u) \]

\[ u' = v \]

\[ v' = -\left( \frac{1}{r} v + \frac{1}{r^2} \partial_{\varphi\varphi} u \right) \]

\[ -D^{-1}(\omega \partial_{\varphi} u + f(u)) \]

\[ r' = 1 \]
The spatial dynamics dictionary

\begin{align*}
-\omega u_\varphi &= D \Delta u + f(u) \quad \leftrightarrow \quad U_r = F(\partial_\varphi, U, r) \\
\lambda v &= D \Delta v + f'(u_{sp}) v \quad \leftrightarrow \quad V_r = A(\partial_\varphi, u_{sp}, r, \lambda)V \\
\text{spiral wave} &\leftrightarrow \text{heteroclinic orbit} \\
\text{linearization} &\leftrightarrow \text{linear bundle} \\
\text{Fredholm properties} &\leftrightarrow \text{hyperbolicity} \\
\text{eigenfunctions} &\leftrightarrow \text{heteroclinic orbits} \\
\text{point spectrum instability} &\leftrightarrow \text{non-transversality} \\
\text{essential spectrum instability} &\leftrightarrow \text{bif’ of periodic orbits at } \infty \\
\cdots &\leftrightarrow \cdots
\end{align*}
Summary: Bifurcations in large domains

- coherent structures: localized effects versus far field
- linear theory: point spectra versus essential and absolute spectra
- period-doubling is a robust wave train doubling in the far field
- nonlinear theory more generally?
- explain slow drift!

Things are not always what they seem to be — but aren’t they pretty?
Acknowledgements and references

• Pictures from BZ-reaction [Park, Lee], [Zhou, Ouyang], [Agladze, Krinsky, Pertsov]

• Period doubling model and experiments

• PD cascades from [A. Libchaber, S. Fauve and C. Laroche, 1983]

• Simulations based on EZSpiral [Barkley] and the Roessler model [Kapral]

• *Period-doubling of spiral waves and defects*, [Sandstede, AS] on www.math.umn.edu/~scheel