(1) Consider the linear equation \( \dot{x} = Ax \), \( A = \text{diag} (\lambda_j) \), \( \lambda_1 > \lambda_2 > \ldots > \lambda_n \).

(a) Derive an equation for the projectivized flow, that is, write \( x = u \cdot |x| \) and derive an equation for \( u \in S^{n-1} \). Find all equilibria of this flow on the sphere.

(b) Show that the Rayleigh quotient \( V(u) = -\frac{1}{2} \langle Au, u \rangle \) is a strict Lyapunov function, that is, strictly decreasing for non-equilibrium solutions. Which equilibria are stable?

(c) Conclude that all trajectories are heteroclinic and describe heteroclinic orbits.

(d) Describe equilibria and heteroclinic orbits for the flow on \( S^1 \) associated with the (non-self-adjoint) \( A = \begin{pmatrix} 0 & 1 \\ \mu & 0 \end{pmatrix} \) for all \( \mu \in \mathbb{R} \)?

(e) Optional supplement:

i. Let \( x_1, \ldots, x_k \) be \( k \) linearly independent vectors in \( \mathbb{R}^n \) and denote by \( E \) the linear subspace spanned by those vectors. Show that the linear equation induces a flow on the set of \( k \)-dimensional subspaces of \( \mathbb{R}^n \).

ii. Show that the space of subspaces is a smooth manifold (the Grassmannian) by locally writing subspaces as graphs: any subspace \( F \) "near" \( E \) can be written as a graph of a linear map \( h(F) : E \mapsto E^\perp \).

iii. Show that the flow on subspaces is a smooth flow, that is, compute the vector field for \( h' = f_A(h) \).

iv. Determine all equilibria of this flow. Find the (unique) stable equilibrium.

(2) Some facts on Lie groups:

(a) Show that \( \Phi_t = e^{At} \) is an orthogonal matrix for all \( t \in \mathbb{R} \), that is \( \Phi_t^{-1} = \Phi_t^T \), if and only if \( A \) is skew symmetric, that is, \( A^T = -A \).

(b) For which matrices \( A \) are all \( e^{At}, t \in \mathbb{R}, \) self-adjoint?

(c) Find a real matrix \( M \in \mathbb{R}^{n \times n} \) such that there does not exist a real matrix \( A \) with \( M = \exp(A) \).

(d) For symmetric or orthogonal \( M \), can you always write \( M = \exp(A) \) for some real \( A \)?

(3) Compute a Jordan normal form transformation and the general solution for the differential equation \( \dot{x} = Ax, \ x(0) = x_0 \), where

\[ \text{The idea is to not use computer algebra but that’s not enforceable.} \]
(a) \[ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

(b) \[ A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

(c) \[ A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

(d) \[ A = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \]

Also try to sketch phase portraits.

(4) We consider the coupled linear system \[ \dot{x} = Ax + D(y - x), \quad \dot{y} = Ay + D(x - y), \]
with \( x, y \in \mathbb{R}^n \), \( D = \text{diag}(d_1, \ldots, d_n) \), and \( d_i > 0 \). We assume throughout that the eigenvalues of \( A \) are contained in the open left complex half plane \( \text{Re}(\text{spec } A) < 0 \).

(a) Note that for initial conditions where \( x(0) = y(0) \), the solution \( (x(t), y(t)) \) converges to zero as \( t \to \infty \).

(b) Find matrices \( A, D \) such that \( x(t) \to \infty \) for suitable initial conditions \( (x(0), y(0)) \) and \( t \to +\infty \).

Try to interpret (see also A. Turing, The chemical basis of morphogenesis, Phil. Trans. Roy. Soc. B 237 (1952), 37–72)

(5) Set \[ A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \]
and consider \[ x' = Ax, \quad x(0) = (1, 0)^T, (1, 1)^T, (1, -1)^T. \]

(a) Find the solutions \( x(t) \) explicitly for the three initial conditions listed.

(b) Solve numerically using and compute \( x(T), t = 1, 3, 5, 10, 15, 20 \); compare with the prediction (try Euler’s method with \( h = 10^{-3} \) and Matlab’s ode45 with tolerance \( 10^{-8} \)).

(c) Take \( x(T) \) as an initial condition for \( x' = -Ax \) and solve for time \( T \), such that the result should be \( x(0) \). Compare the numerical results and explain why and when they drastically differ. How much does this depend on the numerical method?

(d) Find \( \det(e^{At}) \) theoretically. Then find it numerically by computing \( \det(x_1(t)|x_2(t)) \) with initial conditions \( x_1(0) = (1, 0)^T \) and \( x_2(0) = (0, 1)^T \). Show how the numerical determinant agrees for moderate times and then deviates. Why?

Homework is due on Monday, October 29, in class.
For full score, 4 correct exercises!