(1) Consider the feedforward chain,

\[ x'_j = x_{j-1} - x_j \in \mathbb{R}, \]

where at each lattice site \( j \) the system attempts to "mimic" the left neighbor, that is, for fixed \( x_{j-1} \), \( x_j \) will relax to \( x_{j-1} \) (why?).

(a) Take \( 1 \leq j \leq N \) and set \( x_0 \equiv 0 \). Write the system in vector form \( x' = Ax \) and find the eigenvalues. Conclude asymptotic stability.

(b) Take \( 1 \leq j \leq N \) but now set \( x_0 \equiv x_N \) in the equation for \( x_1 \) (the chain is a ring!). Compute the eigenvalues and conclude stability (but not asymptotic stability). \textit{Hint: To find the eigenvalues, use discrete Fourier transform \( x_j = e^{ij\sigma} \) with \( \sigma \) such that \( x_N = x_0 \).}

(c) Solve numerically for large \( N \), \( x_j(t = 0) \equiv 1 \). Demonstrate and explain why the limits \( N \to \infty \) and \( t \to \infty \) do for the linear evolution do not commute, that is,

\[ \lim_{N \to \infty} \lim_{t \to \infty} |e^{At}| = 0, \quad \lim_{N \to \infty} \lim_{t \to \infty} \lim_{N \to \infty} |e^{At}| = 1. \]

(d) Consider an \( \varepsilon \)-feedback, \( x'_1 = \varepsilon x_N - x_1 \). Find the eigenvalues and conclude that for \( \varepsilon \) small fixed, we find arbitrarily slow decay when \( N \) is large. How large does \( N \) have to be to create an eigenvalue \( \lambda = -0.1 \) when \( \varepsilon = 10^{-8} \)?

(2) Construct an example where \( A(t) = A(t + 1) \) is asymptotically stable for every \( t \) but not the non-autonomous equation. \textit{Hint: Try \( A(t) \) piecewise constant, with \( A(t) = A_1, 0 < t < 1/2, \) and \( A(t) = A_2, 1/2 < t < 1, \) and choose \( A_j \) asymptotically stable but not commuting.}

\textit{Homework is due on Wednesday, November 7, in class.}

\textit{This homework counts for extra credit.}

\textit{Those who may, vote before you turn this in!}