(1) Consider a saddle
\[ u' = -u + f(u, v), \quad v' = \rho v + g(u, v), \quad \rho > 0, \]
with \( f, g = O(2) \) and smooth.
Let \((u, v)(t) \to 0\) be a solution in the stable manifold. Show that the solution possesses an exponential expansion, that is,
\[ (u, v)(t) = (u_1, v_1)e^{-t} + (u_2, v_2)e^{-2t} + o(e^{-2t}), \]
(you will see that you can continue to higher orders).
Optional: Compute the coefficients \( u_j, v_j \) assuming that \( u_1 = \delta \ll 1 \) to order \( \delta^2 \), using the Taylor expansion of \( f, g \).

(2) Recall the amplitude equations for rotating convection
\[ x' = x(1 - x^2 - by^2 - cz^2), \]
\[ y' = y(1 - y^2 - bz^2 - cx^2), \]
\[ z' = z(1 - z^2 - bx^2 - cy^2). \]

(a) Find all equilibria, determine their linear stability, and their nonlinear stability when the linearization is hyperbolic.
(b) Find the parameters \( b, c \) such that there are no equilibria with \( x = 0, yz \neq 0 \) and such that all equilibria are unstable.
(c) Simulate the case \( c = 3, b = 0 \) in the octant \( x, y, x \geq 0 \). What do you see in the coordinate planes? What happens to solutions in \( x, y, z > 0 \)?
(d) Show that in the parameter regime (b) (or, specifically, for the parameters in (c)), the \( \omega \)-limit set of a solution in \( x > y > z > 0 \) cannot be a single equilibrium.

(3) Consider \( u' = -u + v, \quad v' = -v + u^2 \). Show that there exists a one-dimensional, smooth invariant manifold tangent to the eigenspace. Compute its quadratic expansion. Hint: Use projection coordinates \( v_1 = v/u, u_1 = u \) and find the strong stable manifold of the origin in the new coordinates.

(4) Consider \( x' = -x, y' = -2y + x^2 \). The linear equation possesses a weak stable subspace \( E^s = \{ y = 0 \} \).

(a) Assume that there exists a \( C^2 \)-invariant manifold tangent to this subspace and try to compute its quadratic terms at the origin: what goes wrong?
(b) Find solutions explicitly and show that manifolds tangent to $E^s$ are not of class $C^2$.

*Homework is due on Monday, November 19, in class.*

*For full score, 4 correct exercises!*