Math 8502, Spring 2019

Dynamical Systems and Differential Equations II

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— *Homework 1* —

(1) We consider the extended system $G : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ for locating turning points of $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$,

$$G(x, y, \mu) := \begin{pmatrix} f(x,\mu) \\ D_x f(x,\mu) \cdot y \\ (y, y) - 1 \end{pmatrix} = 0,$$

with $x, y \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$. Suppose that the linearization $D_{x,y,\mu}G$ is invertible at a zero (x, y, μ). Conclude that the solution branch to $f(x, \mu)$ undergoes a non-degenerate saddle-node bifurcation at this point.

(2) Use the function G from (1) to design a continuation code for saddle-node bifurcations in the two-parameter family $f(x, \nu) = 0$, $\nu \in \mathbb{R}^2$ using arclength continuation. Implement the strategy to continue saddle-node bifurcations in the parameters $\nu = (d, a)$ in

$$0 = F(u_1, u_2, d, a) := \begin{pmatrix} d(u_2 - u_1) + u_1(1 - u_1)(u_1 - 1/2) + a \\ d(u_1 - u_2) + u_2(1 - u_2)(u_2 - 1/2) + a \end{pmatrix}.$$

Submit the code and the result of the continuation of the saddle-node on the branch $u_1 = 1, u_0 = 0$, $a = 0, d = 0$, that is, continue from this starting value using the strategy in class to find a saddle-node. Continue this saddle-node with the approach developed here.

(3) Examine the perturbed "double" saddle-node for equilibria of

$$x' = \mu + x^2 + \varepsilon(y - x), \qquad y' = \mu + y^2 + \varepsilon(x - y),$$

with $\mu, \varepsilon \sim 0$. Find (analytically) all solutions and determine the bifurcations as μ varies and $\varepsilon > 0$, small. Draw bifurcation diagrams plotting $x + y$ versus μ and $x - y$ versus μ for $\varepsilon > 0$, small and for $\varepsilon < 0$, small. For each branch, determine stability, that is, compute the number of unstable eigenvalues of the linearization.

(4) Use arclength continuation to find all solutions branches to

$$0 = F(u_1, u_2, d, a) := \begin{pmatrix} d(u_2 - u_1) + u_1(1 - u_1)(u_1 - 1/2) + a \\ d(u_1 - u_2) + u_2(1 - u_2)(u_2 - 1/2) + a + \varepsilon \end{pmatrix}.$$

with $d = 0.01$, in the 3 cases $\varepsilon = 0, \pm 0.01$. Describe how the bifurcation diagram changes and explain how the change reflects Sard's theorem. What is the difference between $\varepsilon = +0.01$ and $\varepsilon = -0.01$?

Four points each exercise. Homework is due on Wednesday, February 5, in class.