(1) Consider the bifurcation problem for the ODE $A \in \mathbb{C} \sim \mathbb{R}^2$, 

$$A' = (\mu^2 + i)A + \mu(1 + i)A|A|^2 + A\bar{A}^2 + iA^3.$$ 

We wish to find all branches of periodic solutions bifurcating from the origin.

(i) Consider the equation at $\mu = 0$, $A' = iA + \bar{A}A^2 + iA^3$ and perform a normal form change of variables $A = A_1 + \alpha_1 A_1 \bar{A}_1^2 + \alpha_2 A_1^3$. Find $\alpha_j$ so that cubic terms vanish in the resulting equation.

(ii) Argue that a second normal form coordinate change will eliminate all quintic terms except for a term of the form $\beta A|A|^4$ for some $\beta \in \mathbb{C}$. Find $\beta = -(1+i)!$

(iii) Study the resulting leading-order system 

$$A' = (\mu^2 + i)A + \mu(1 + i)A|A|^2 - A|A|^4,$$

using an Ansatz $A = re^{i\omega t}$ using Newton’s polygon for variables $r, \mu$ in the real part of the equation to determine all bifurcating branches.

(iv) Find expansions for $r$ and $\omega$ in terms of $\mu$ for periodic orbits.

(2) The equation 

$$b' = v - sb, \hspace{1cm} v' = -b^2(\theta - v) + b$$

with parameters $s, \theta$ arises in a model for traveling waves in vegetation patterns. In the parameter plane $\theta, s \geq 0$, plot curves of saddle-node bifurcations and mark regions with 1, 2, or 3 equilibria. Also find all Hopf bifurcations and show that the curve of Hopf bifurcations in the $\theta - s$-plane terminates on the saddle-node curve in a Bogdanov-Takens bifurcation.

Optional: Compute the direction of branching of the Hopf bifurcation, that is, if the Hopf bifurcation is sub- or supercritical in the parameter $\theta$ for fixed $s$. You may use a formula for the cubic coefficient from the literature and Mathematica to evaluate derivatives.
(3) Consider
\[ z' = -2z + (A^2 + \bar{A}^2) + z^2, \quad A' = iA - 3iz\bar{A}, \]
with \( A \in \mathbb{C}, z \in \mathbb{R}, \) near \( A = z = 0. \) Determine if the origin is asymptotically stable. For this, compute the center manifold to quadratic order, \( z = h(A, \bar{A}), \) and transform the reduced equation for \( A \) into cubic normal form to determine if the origin is stable within the center manifold.

*Four points each exercise. Homework is due on Wednesday, February 27, in class.*