Math 8502, Spring 2019

Dynamical Systems and Differential Equations II

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— *Homework 2* —

(1) Consider the bifurcation problem for the ODE $A \in \mathbb{C} \sim \mathbb{R}^2$,

$$A' = (\mu^2 + \mathrm{i})A + \mu(1 + \mathrm{i})A|A|^2 + A\bar{A}^2 + \mathrm{i}A^3.$$

We wish to find all branches of periodic solutions bifurcating from the origin.

(i) Consider the equation at $\mu = 0$, $A' = \mathrm{i}A + A\bar{A}^2 + \mathrm{i}A^3$ and perform a normal form change of variables $A = A_1 + \alpha_1 A\bar{A}_1^2 + \alpha_2 A_1^3$. Find α_j so that cubic terms vanish in the resulting equation.

(ii) Argue that a second normal form coordinate change will eliminate all quintic terms except for a term of the form $\beta A|A|^4$ for some $\beta \in \mathbb{C}$. Find $\beta = -(1+\mathrm{i})$!

(iii) Study the resulting leading-order system

$$A' = (\mu^2 + \mathrm{i})A + \mu(1 + \mathrm{i})A|A|^2 - A|A^4,$$

using an Ansatz $A = r\mathrm{e}^{\mathrm{i}\omega t}$ using Newton's polygon for variables r, μ in the real part of the equation to determine all bifurcating branches.

(iv) Find expansions for r and ω in terms of μ for periodic orbits.

(2) The equation

$$b' = v - sb,$$
$$v' = -b^2(\theta - v) + b$$

with parameters s, θ arises in a model for traveling waves in vegetation patterns. In the parameter plane $\theta, s \geq 0$, plot curves of saddle-node bifurcations and mark regions with 1,2, or 3 equilibria. Also find all Hopf bifurcations and show that the curve of Hopf bifurcations in the $\theta - s$-plane terminates on the saddle-node curve in a Bogdanov-Takens bifurcation.

Optional: Compute the direction of branching of the Hopf bifurcation, that is, if the Hopf bifurcation is sub- or supercritical in the parameter θ for fixed s. You may use a formula for the cubic coefficient from the literature and Mathematica to evaluate derivatives.

(3) Consider

$$z' = -2z + (A^2 + \bar{A}^2) + z^2, \qquad A' = \mathrm{i}A - 3\mathrm{i}z\bar{A},$$

with $A \in \mathbb{C}, z \in \mathbb{R}$, near $A = z = 0$. Determine if the origin is asymptotically stable. For this, compute the center manifold to quadratic order, $z = h(A, \bar{A})$, and transform the reduced equation for A into cubic normal form to determine if the origin is stable within the center manifold.

Four points each exercise. Homework is due on Wednesday, February 27, in class.