Math 8502, Spring 2019

Dynamical Systems and Differential Equations II

Arnd Scheel, VinH 509, phone 625-4065, scheel@umn.edu

— *Homework 3* —

(1) *"Arnol'd normal form for matrices":* Let $A(\mu)$ be a smooth one-parameter family of matrices such that $A(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We want to show that there exists a family of matrices $T(\mu)$, $T(0) = \mathrm{id}$, such that $A_{\mathrm{nf}}(\mu) = T^{-1}(\mu)A(\mu)T(\mu) = \begin{pmatrix} a_1(\mu) & 1 \\ a_2(\mu) & a_1(\mu) \end{pmatrix}$.

(i) Write $T(\mu) = \mathrm{id} + \mu T_1(\mu)$ and show that $A_{\mathrm{nf}}(\mu) = A(\mu) + \mu[A(0), T_1] + \mathrm{O}(\mu^2)$, where $[A, B] := AB - BA$.

(ii) Determine the range of the linear map $\mathrm{ad}_1(A) : T_1 \mapsto [A, T_1]$ as the matrices of the form $\begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}$.

(iii) Conclude that the set of matrices $\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ form a complement to the range and that we can find T_1 such that A_{nf} is in the desired normal form at order μ.

(iv) Find the normal form using the implicit function theorem with parameter μ.

(v) Show that the range of $\mathrm{ad}_1(A)$ is perpendicular to the kernel of $\mathrm{ad}_1(A^T)$when using the scalar product on matrices $A \cdot B := \mathrm{tr}(AB^T)$.

The considerations here easily generalize to length-n Jordan blocks; see *V.I. Arnold, On matrices depending on parameters, Russian Math. Surveys **26** (2) (1971) 29–43.*

(2) *"Renormalizing cirlce rotations and continued fractions":* Consider the rigid circle rotation $R_\alpha : x \mapsto x + \alpha \mod 1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $J_0 = [0, 1 - \alpha)$ and $J_1 = [1 - \alpha, 1)$.

(i) Construct the first return map $\Phi : J_1 \to J_1$, that is, $\Phi(x) = R_\alpha^{k(x)}(x)$ with $k(x) > 0$ minimal so that $\Phi(x) \in J_1$. Show that $k(x) \in \{\ell, \ell + 1\}$, where $\ell = \left[\frac{1}{\alpha}\right]$, and $[z]$ denotes the integer part of z.

(ii) Show that $\Phi(1 - \alpha) = \lim_{x \to 1} \Phi(x)$, so that Φ defines a continuous circle map on the "small" circle $[1 - \alpha, 1]/(1 - \alpha \sim 1)$.

(iii) Let $\Psi : [1 - \alpha, 1) \to (0, 1], x \mapsto y = \frac{1-x}{\alpha}$ be the unique affine, orientation reversing map from the "small" circle to the standard circle. Show that $\Psi \circ \Phi \circ \Psi^{-1}$ is a rigid rotation R_β of S^1 and compute β.

(iv) One can clearly iterate this procedure. Interpret the resulting return times ℓ as denominators in the continued fraction expansion of α.

(v) Consider an (arbitrary) orbit $x_j = x + j \cdot \alpha \mod 1$. Define a coding sequence $a_j \in \{0, 1\}$ so that $a_j = k$ if $x_j \in J_k$. Show that a_j consists of ℓ or $\ell - 1$ 0's followed by precisely one 1. What is ℓ?

(vi) Define a renormalized sequence b_j, as replacing a block of ℓ 0's and a following 1 by a 0, a block of $\ell - 1$ 0's following a 1 by 1. Show that the renormalized sequence is of the same type as the first sequence. Explain the relation to the previous considerations of renormalized return maps.

Six points each exercise. Choose one! Choose two for extra credit. Homework is due on Friday, March 8, in class.