(1) “Arnol’d normal form for matrices”: Let $A(\mu)$ be a smooth one-parameter family of matrices such that $A(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We want to show that there exists a family of matrices $T(\mu)$, $T(0) = \text{id}$, such that $A_{nf}(\mu) = T^{-1}(\mu)A(\mu)T(\mu) = \begin{pmatrix} a_1(\mu) & 1 \\ a_2(\mu) & a_1(\mu) \end{pmatrix}$.

(i) Write $T(\mu) = \text{id} + \mu T_1(\mu)$ and show that $A_{nf}(\mu) = A(\mu) + \mu [A(0), T_1] + O(\mu^2)$, where $[A, B] := AB - BA$.

(ii) Determine the range of the linear map $\text{ad}_1(A) : T_1 \mapsto [A, T_1]$ as the matrices of the form $\begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}$.

(iii) Conclude that the set of matrices $\begin{pmatrix} a & 0 \\ b & a \end{pmatrix}$ form a complement to the range and that we can find $T_1$ such that $A_{nf}$ is in the desired normal form at order $\mu$.

(iv) Find the normal form using the implicit function theorem with parameter $\mu$.

(v) Show that the range of $\text{ad}_1(A)$ is perpendicular to the kernel of $\text{ad}_1(A^T)$ when using the scalar product on matrices $A \cdot B := \text{tr}(AB^T)$.


(2) “Renormalizing circle rotations and continued fractions”: Consider the rigid circle rotation $R_\alpha : x \mapsto x + \alpha \mod 1$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Let $J_0 = [0, 1 - \alpha)$ and $J_1 = [1 - \alpha, 1)$.

(i) Construct the first return map $\Phi : J_1 \to J_1$, that is, $\Phi(x) = R_\alpha^{k(x)}(x)$ with $k(x) > 0$ minimal so that $\Phi(x) \in J_1$. Show that $k(x) \in \{\ell, \ell + 1\}$, where $\ell = \lfloor \frac{1}{\alpha} \rfloor$, and $\lfloor z \rfloor$ denotes the integer part of $z$.

(ii) Show that $\Phi(1 - \alpha) = \lim_{x \to 1} \Phi(x)$, so that $\Phi$ defines a continuous circle map on the “small” circle $[1 - \alpha, 1]/(1 - \alpha \sim 1)$.

(iii) Let $\Psi : [1 - \alpha, 1) \to (0, 1], x \mapsto y = \frac{1-x}{\alpha}$ be the unique affine, orientation reversing map from the “small” circle to the standard circle. Show that $\Psi \circ \Phi \circ \Psi^{-1}$ is a rigid rotation $R_\beta$ of $S^1$ and compute $\beta$.

(iv) One can clearly iterate this procedure. Interpret the resulting return times $\ell$ as denominators in the continued fraction expansion of $\alpha$. 
(v) Consider an (arbitrary) orbit $x_j = x + j \cdot \alpha \mod 1$. Define a coding sequence $a_j \in \{0, 1\}$ so that $a_j = k$ if $x_j \in J_k$. Show that $a_j$ consists of $\ell$ or $\ell - 1$ 0’s followed by precisely one 1. What is $\ell$?

(vi) Define a renormalized sequence $b_j$, as replacing a block of $\ell$ 0’s and a following 1 by a 0, a block of $\ell - 1$ 0’s following a 1 by 1. Show that the renormalized sequence is of the same type as the first sequence. Explain the relation to the previous considerations of renormalized return maps.

*Six points each exercise. Choose one! Choose two for extra credit. Homework is due on Friday, March 8, in class.*