The proof of the Seifert-van Kampen theorem

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Proof. This proof is pretty long and involved, and we might sketch over some parts.

First let’s give names to things. Let \( i_U : U \to X, i_V : V \to X, j_U : U \cap V \to U \) and \( j_V : U \cap V \to V \) be the inclusion maps. Then the claim is that given any group \( K \) and a commutative diagram

\[
\begin{array}{ccc}
\pi_1(U \cap V, x_0) & \xrightarrow{j_U \ast} & \pi_1(U, x_0) \\
\downarrow{j_V \ast} & & \downarrow{\phi} \\
\pi_1(V, x_0) & \xrightarrow{\psi} & K,
\end{array}
\]

there is a unique homomorphism \( \lambda : \pi_1(X, x_0) \to K \) such that the diagram

\[
\begin{array}{ccc}
\pi_1(U \cap V, x_0) & \xrightarrow{j_U \ast} & \pi_1(U, x_0) \\
\downarrow{j_V \ast} & & \downarrow{\phi} \\
\pi_1(V, x_0) & \xrightarrow{i_U \ast} & \pi_1(X, x_0) \\
& & \downarrow{\lambda} \\
& & K \\
& & \downarrow{\psi}
\end{array}
\]

commutes. Then \( \pi_1(X, x_0) \) will satisfy the mapping property of the free product with amalgamation, so it must be isomorphic to the free product with amalgamation.

We’ll tackle uniqueness of \( \lambda \) first, then existence. For uniqueness, it will suffice to show that \( \pi_1(X, x_0) \) is generated by im \( (i_U \ast) \) and im \( (i_V \ast) \) (this means that the smallest subgroup of \( \pi_1(X, x_0) \) containing the two images is \( \pi_1(X, x_0) \) itself, or equivalently, that any element of \( \pi_1(X, x_0) \) can be written as a product of elements of the two images.) Then if \( \lambda \) exists, it must be unique, since

\[
\lambda(i_U \ast (\gamma)) = \phi(\gamma), \lambda(i_V \ast (\delta)) = \psi(\delta).
\]

So let’s prove this:

Theorem 1. In the situation of the Seifert-van Kampen theorem, \( \pi_1(X, x_0) \) is generated by im \( (i_U \ast) \) and im \( (i_V \ast) \).
Proof. This is actually not that hard. Let \( f : [0, 1] \to X \) be a path with \( f(0) = f(1) = x_0 \). By an argument we’ve seen before with the Lebesgue number lemma, we can find \( N \in \mathbb{N} \) such that for each \( i = \{0, 1, \cdots, N - 1\} \), either \( f([i/N, i/N + 1]) \in U \) or \( f([i/N, i/N + 1]) \in V \).

Now let
\[
A = \{ a \in \left\{0, \frac{1}{N}, \cdots, \frac{N - 1}{N}, 1\right\} \mid f(a) \in U \cap V \}
= \{0 = a_0, a_1, \cdots, a_{m-1}, a_m = 1\}.
\]

Then for each \( i \) with \( 0 \leq i \leq m \), it’s still the case that \( f[a_i, a_{i+1}] \subseteq U \) or \( f[a_i, a_{i+1}] \subseteq V \). Indeed, \( f[a_i, a_{i+1}] \) is the concatenation of several segments with this property, but if it switched between \( U \) and \( V \) at some point, one of the stopping points we removed would have to be in \( U \cap V \). But those are the ones we kept. Define \( f_i : [0, 1] \to X \) by
\[
f_i(t) = (a_i + (a_{i+1} - a_i)t).
\]

In other words, it’s just the path \( f \) takes from \( f(a_i) \) to \( f(a_{i+1}) \), linearly reparametrized so that its domain is \( [0, 1] \). (Draw a picture.) So
\[
[f] = [f_0] \ast [f_1] \ast \cdots \ast [f_{m-1}].
\]

Now we crucially use the fact that \( U \cap V \) is path-connected. For each \( i \) with \( 0 < i < m \), let \( \alpha_i \) be a path from \( x_0 \) to \( a_i \) in \( U \cap V \). Then define
\[
g_i = \alpha_i \ast f_i \ast \alpha_{i+1}^{-1}.
\]

For each \( i \), either \( g_i \in \pi_1(U, x_0) \) or \( \pi_1(V, x_0) \) (depending on \( f_i \)). But
\[
[f] = [g_0] \ast [g_1] \ast \cdots \ast [g_{m-1}].
\]

This proves the theorem, and thus uniqueness of \( \lambda \).

This already buys us something:

**Corollary 2.** Suppose that \( U, V \) are open subsets of \( X \) such that \( X = U \cup V \), \( U \cap V \) is path-connected and \( U \) and \( V \) are both simply connected. Then \( X \) is simply connected.

**Proof.** Clearly \( X \) is path connected. Now by the previous result, we know that \( \pi_1(X, x_0) \) is generated by \( \text{im} iv_* \) and \( \text{im} iv_* \). But these are homomorphisms from trivial groups, so their images are just the identity. Thus \( \pi_1(X, x_0) \) must be the trivial group.

**Corollary 3.** \( S^n \) is simply connected for \( n \geq 2 \).

**Proof.** Let \( x_N, x_S \) denote the north pole and the south pole in \( S^n \). Let \( U = S^n \setminus \{x_N\} \) and \( V = S^n \setminus \{x_S\} \). Then \( U \) and \( V \) are each contractible, so certainly simply connected, and \( U \cap V \) is path-connected. So \( S^n \) is simply connected.

Note this fails for \( n = 1 \), since in that case \( U \cap V \) fails to be path-connected.
Now let’s do the harder part of the proof: showing the existence of $\lambda$. I’m going to basically read from the book on this one (p.426 - 430), so I’m not going to copy it down here.