Thanks to the organizers for inviting me to speak. Convention: everything in this talk will be homotopy invariant, so all of my categories will be $\infty$-categories, all of my colimits homotopy colimits, and so on. I’ll begin by giving rapid-fire refreshers on both functor calculus and equivariant stable homotopy theory.

1 Funct or calculus

All results, unless otherwise stated, are due to Goodwillie. Let $\mathbf{Sp}$ be the category of spectra and let $F : \mathbf{Sp} \to \mathbf{Sp}$ be a functor. All of our functors will be reduced: that is, $F(*) = *$.

An $n$-cube of spectra will be called strongly cocartesian if it’s constructed by writing down $n$ morphisms $X \to Y_1, X \to Y_2, \cdots, X \to Y_n$ and then taking iterated pushouts. (Example with 3-cube.) $F$ will be called $n$-excisive if it takes strongly cocartesian $n$-cubes to cartesian $n$-cubes - cubes which are limit diagrams, or equivalently (since we’re stable) colimit diagrams.

Any functor has a right universal $n$-excisive approximation $F \to P_n F$. $F$ will be called $n$-homogeneous if $F = P_n F$ and $P_{n-1} F = 0$.

**Proposition 1.1.** $F$ is $n$-homogeneous if and only if it’s equivalent to a functor of the form $F(X) = (D \wedge X^n)_{h \Sigma_n}$ for some spectrum $D$ with $\Sigma_n$-action.

$D$ is supposed to look like a Taylor coefficient; this formula is a categorification of

$$f(x) = \frac{d x^n}{n!}.$$  

For example, the fiber of $P_n F \to P_{n-1} F$ is $n$-homogeneous. The coefficient spectrum $D_n$ arising here is called the $n$th derivative of $F$. So we have a fiber sequence

$$(D_n \wedge (-)^n)_{h \Sigma_n} \to P_n F \to P_{n-1} F.$$  

Stratified categories, geometric fixed points and a generalized Arone-Ching theorem: talk notes

Saul Glasman
In particular, suppose \( F \) is 2-excisive. Then \( P_1F \) is 1-homogeneous, and so we have a fiber sequence

\[
(D_2 \wedge X \wedge X)_{\wedge \Sigma_2} \to F(X) \to D_1 \wedge X.
\]

The last general thing I want to say about calculus is this proposition - I’m not sure who to credit with this, but it’s well-known.

**Proposition 1.2.** If \( F \) is \( n \)-excisive, then it’s determined by its restriction to the full subcategory of \( \text{Sp} \) spanned by the wedges of up to \( n \) copies of the sphere spectrum \( \Sigma \).

This suggests that \( n \)-excisive functors might be somehow combinatorial at heart. We’ll return to this.

## 2 Equivariant homotopy theory

Let \( G \) be a finite group. The first realization in equivariant stable homotopy theory is that a \( G \)-spectrum is much more than a spectrum with a \( G \)-action. There are several models of what we call genuine \( G \)-spectra, but in this talk by a \( G \)-spectrum we’ll mean a spectral Mackey functor, which I’ll now explain.

Let \( C \) be a category with pullbacks. Then we can form the effective Burnside category of \( C \), denoted \( A_{\text{eff}}(C) \), whose objects are those of \( C \) and in which morphisms are spans (draw). Compositions are defined by forming pullbacks.

If \( C \) is disjunctive - that is, it has finite coproducts and pullbacks distribute over them - then \( A_{\text{eff}}(C) \) is semiadditive - that is, it has a zero object and finite direct sums, so that finite coproducts coincide with finite products and are given by the coproduct in \( C \).

**Definition 2.1.** A Mackey functor on \( C \) is a direct-sum-preserving functor from \( A_{\text{eff}}(C) \) to \( \text{Sp} \).

**Definition 2.2.** A \( G \)-spectrum is a Mackey functor on the category \( F^G \) of finite \( G \)-sets.

What does this amount to? Let’s restrict attention to the irreducible \( G \)-sets of the form \( G/H \), where \( H \) is a subgroup of \( G \). Then a \( G \)-spectrum \( E \) gives a spectrum \( E^H \) - the “genuine fixed points” - for each \( H \), which has an appropriate group action, and whenever \( H \leq K \), we get a restriction map \( E^K \to E^H \) and a transfer map \( E^H \to E^K \).

This picture is adapted to dealing with the genuine fixed points of spectra, which are both a left and a right adjoint and which are compatible with \( \Omega^\infty \) but not as easily related to \( \Sigma^\infty \). There’s another notion of fixed points - the geometric fixed points \( E \mapsto \Phi^H E \) - which are a left adjoint only, but play well with the formation of suspension spectra.

Here’s an important example of a \( G \)-spectrum. Let \( A \) be a spectrum, and form \( A^G \): the smash product of copies of \( A \) indexed by \( G \), with \( G \) permuting
the factors, also known as the “HHR norm”. Making this into a genuine $G$-spectrum is surprisingly subtle - one can construct it at the point set level, and then proving homotopy invariance is a nightmare, but from the Mackey functor point of view it’s not even how to start. This is because the genuine fixed point spectrum $(A^G)^G$ is full of mysteries; for example, it’s intimately connected to Witt vectors, which is something I’m still amazed by every day. I mean, all you did was take some fixed points.

On the other hand, we have

$$\Phi^G(A^G) \simeq A.$$  

This raises the question: can we present a $G$-spectrum in terms of its geometric fixed point spectra?

## 3 The dictionary

Now suppose that $G = C_2$. Then there is a “norm cofibration sequence”, due to Greenlees and May:

$$E_{hG} \rightarrow E^G \rightarrow \Phi^G E.$$  

In particular, if $E$ is the norm $A \wedge A$, then this sequence takes the form

$$(A \wedge A)_{hC_2} \rightarrow (A \wedge A)^{C_2} \rightarrow \Phi^{C_2}(A \wedge A) \simeq A.$$  

Strikingly, this sequence coincides with the Taylor tower for the 2-excisive functor $A \mapsto (A \wedge A)^{C_2}$. We see that the first and second derivatives of this functor are both $\mathbb{S}$. In fact, this is the motivating case for the following even more striking theorem:

**Theorem 3.1** (?). There is an equivalence between the category of $C_2$-spectra and the category of 2-excisive functors which takes a $C_2$-spectrum $D$ to the functor

$$A \mapsto (A \wedge A \wedge X)^{C_2}.$$  

This theorem establishes a dictionary between equivariance and calculus, which I’ll deliberately write without reference to $C_2$ or the number 2:

<table>
<thead>
<tr>
<th><strong>$G$-spectrum $E$</strong></th>
<th><strong>$n$-excisive functor $F$</strong></th>
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</thead>
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<tr>
<td>Norm cofibration sequence and its generalizations</td>
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Challenge: can we come up with a framework that incorporates both these phenomena?
4 Epiorbital categories

The answer is yes, because \( n \)-excisive functors are Mackey functors too. Before I say exactly how, I want to fix a nice class of indexing categories:

**Definition 4.1.** A category is **epiorbital** if

1. it’s an essentially finite 1-category, i.e. it has finitely many isomorphism classes of objects and finite hom-sets;
2. all morphisms are epimorphisms;
3. pushouts and coequalizers exist.

The first two conditions imply that if \( X \) and \( Y \) are two objects and morphisms between them exist in each direction, then those morphisms are isomorphisms. So an epiorbital category should be thought of as a “poset with automorphisms”.

The most important examples for the purposes of this talk are the orbit category \( \mathcal{O}_G \) of a finite group \( G \) (i.e. irreducible \( G \)-sets) and the category \( \mathcal{F}_{\leq n} \) of finite sets of cardinality at most \( n \) and surjections.

**Proposition 4.2.** If \( C \) is an epiorbital category then the category \( C^{\Pi} \) obtained by freely adjoining finite coproducts to \( C \) is disjunctive. (Remember that disjunctive categories are the ones whose categories of spans behave nicely.)

**Pretheorem 4.3 (G.).** The category of \( n \)-excisive functors \( \mathbf{Sp} \to \mathbf{Sp} \) is equivalent to the category of Mackey functors on \( (\mathcal{F}_{\leq n})^{\Pi} \). If \( F \in \text{Fun}^{n-\text{exc}}(\mathbf{Sp}, \mathbf{Sp}) \), the value of the corresponding Mackey functor on \( S \) is the cross effect

\[
\text{cr}_S F(S, S, \cdots, S).
\]

This has pretheorem status in that I know it’s true and I know how to prove it, but I have to nail down some technicalities about semiadditive \( \infty \)-categories. Note that this is a topological version of an algebraic theorem by Baues, Dreckmann, Franjou and Pirashvili.

What’s the analog of geometric fixed points for Mackey functors on an arbitrary epiorbital category? Let \( C \) be epiorbital and \( X \in C \). Let \( C^{\leq X} \) be the full subcategory of \( C \) spanned by the objects admitting a map from \( X \); these are all different for nonisomorphic \( X \). There’s a functor

\[
j_X : C^{\Pi} \to (C^{\leq X})^{\Pi}
\]

given by setting all objects outside \( C^{\leq X} \) to the empty set; then the “geometric fixed points at \( X \)” are given by left Kan extension along \( j_X \). This recovers ordinary geometric fixed points in the case of \( G \)-spectra and derivatives in the case of \( n \)-excisive functors.
5 The reconstruction theorem

Now we’ll investigate how to reconstruct Mackey functors on epiorbital categories from their geometric fixed point spectra. For motivation, let’s first return to the case of $C_2$-spectra. We have a map of cofiber sequences

$$
\begin{array}{ccc}
E_{hC_2} & \rightarrow & E^{C_2} \\
\downarrow & & \downarrow \\
E_{hC_2} & \rightarrow & E^{hC_2}
\end{array}
\rightarrow
\begin{array}{ccc}
\Phi C_2 E & \rightarrow & \Phi^{C_2} E \\
\downarrow & & \downarrow \\
E_{hC_2} & \rightarrow & E^{hC_2}
\end{array}
\rightarrow
\begin{array}{ccc}
E_{hC_2} & \rightarrow & E^{hC_2}
\end{array}

where the bottom cofiber sequence is the definition of the Tate spectrum $E^{tC_2}$. Thus the right hand square is a pullback square, and we learn that the glue needed to reconstruct $E^{C_2}$ from the geometric fixed point spectra of $E$ is the map from $\Phi^{C_2} E$ to $E^{tC_2}$. In previous work on this subject, Abram and Kriz have generalized this “fracture square” to $G$-spectra for abelian $G$, and Arone and Ching have given an analog for $n$-excisive functors. Our common generalization is as follows:

**Theorem 5.1** (G.). Let $M$ be a Mackey functor on an epiorbital category $C$. Then there is an explicit cartesian (Iso $C$)-cube with limit $M$ whose other vertices are given by homotopy-theoretic constructions (homotopy orbits and fixed points, Tate spectra) on geometric fixed point spectra of $M$.

OK, that’s a tangle. Let’s see it in action. We’ve seen the case where $C = O_{C_2}$, and $C_p$ for prime $p$ is exactly the same. Let’s take $O_{C_{p^2}}$ this time. For convenience, I’ll evaluate the entire cube of Mackey functors on $G/G$.

\[
\begin{array}{ccc}
X^{hC_{p^2}} & \rightarrow & (X^{tC_{p^2}})^{hC_{p^2}} \\
\downarrow & & \downarrow \\
(X^{C_{p^2}})^{hC_{p^2}} & \rightarrow & (\Phi^{C_{p^2}} X)^{hC_{p^2}}
\end{array}
\rightarrow
\begin{array}{ccc}
X^{hC_{p^2}} & \rightarrow & (X^{tC_{p^2}})^{hC_{p^2}} \\
\downarrow & & \downarrow \\
(X^{C_{p^2}})^{hC_{p^2}} & \rightarrow & (\Phi^{C_{p^2}} X)^{tC_{p^2}}
\end{array}
\rightarrow
\begin{array}{ccc}
0
\end{array}
\]

The $(X^{tC_{p^2}})^{hC_{p^2}}$ entry requires a little computation. The deepest entry turns out to be 0, and it’s not unusual for all entries at depth at least 3 to be 0.

6 Chromatic theory

Fix a prime $p$ and let $K(n)$ be the $n$th Morava $K$-theory. It’s a deep theorem coming from I guess work of Greenlees-Sadofsky and Hovey-Sadofsky that Tate
spectra all vanish in the category of $K(n)$-local spectra. Nick Kuhn used this to show that Taylor towers all split in the $K(n)$-local category.

Here’s a general theorem along those lines:

**Proposition 6.1** (G.). Every entry of the cube of depth at least 2 contains a Tate construction, and so vanishes in the $K(n)$-local category. This gives a product decomposition of any Mackey functor valued in $K(n)$-local spectra. In the case of calculus, this gives Kuhn’s splitting; for $G$-spectra, it gives a version of the tom Dieck splitting.

### 7 Stratified $\infty$-categories

Let’s say a couple of words about the proof of the main theorem. In fact, the category $\text{Mack}(\mathbf{C}^{hl})$ for an EOC $\mathbf{C}$ is an example of an abstract structure known as a stratified stable $\infty$-category:

**Definition 7.1.** Let $\mathcal{P}$ be a finite poset and let $\mathcal{I}_\mathcal{P}$ be the poset of intervals in $\mathcal{P}$, ordered by inclusion. A stratification of a stable $\infty$-category $\mathbf{D}$ along a finite poset $\mathcal{P}$ is an exact localization $L_I$ for each interval $I \subseteq \mathcal{P}$ such that

1. for $I_1 \subseteq I_2$, $I_2$-local implies $I_1$-local, and
2. whenever $I$ and $J$ are intervals such that there are no $j \in J$ and $i \in I$ with $j > i$, the natural diagram

$$
\begin{align*}
L_{I \cup J} &\longrightarrow L_I \\
\downarrow &\downarrow \\
L_J &\longrightarrow L_J L_I
\end{align*}
$$

is a pullback diagram of functors.

Then there’s a general reconstruction theorem that expresses any object of $\mathbf{D}$ as a limit of objects which belong to $L_{\{p\}} \mathbf{D}$ for elements $p \in \mathcal{P}$. So your minimal intervals.

For $\text{Mack}(\mathbf{C}^{hl})$, the poset $\mathcal{P}$ is the underlying poset of $\mathbf{C}$. For an interval $I$ which is downwards-closed, the $I$-local objects are those Mackey functors which are supported on the objects in $I$, and for general intervals, it’s something slightly more complicated that I can tell you about if you’re interested.

There are other examples of stratified categories too: for instance, the category of $E_n$-local spectra is stratified by localizations at wedges of Morava $K$-theories, and our reconstruction theorem applies there too.