Problem 1  [7pt] Evaluate \( \int \frac{1}{\sqrt{3+2x-x^2}} \, dx \)

First we rewrite the quadratic term inside the square root as complete square: \( 3+2x-x^2 = 4-(x-1)^2 \).

To do this,
First, pull out the coefficient of the highest degree term: \( 3 + 2x - x^2 = -(x^2 - 2x - 3) \).

Second, look at the coefficient of \( x \), which is \(-2\). Divide it by 2 and you have \(-1\). This means you can write \( x^2 - 2x - 3 \) as \( (x-1)^2 + C \), where \( C \) is unknown constant.

Third, find the value of \( C \) by comparing \( x^2 - 2x - 3 \) and \( (x-1)^2 + C \). Since \( x^2 - 2x - 3 = (x-1)^2 + C = x^2 - 2x + 1 + C \), we know \( C = -4 \).

So, \( 3+2x-x^2 = -(x^2 - 2x - 3) = -(x-1)^2 - 4 = 4 - (x-1)^2 \).

\[
\int \frac{1}{\sqrt{3+2x-x^2}} \, dx = \int \frac{1}{\sqrt{4-(x-1)^2}} \, dx.
\]

Method 1 Trigonometric Substitution:
As we can see the integral is of the form \( \sqrt{a^2-u^2} \), we let \( x-1 = 2\sin \theta \).

\[
\int \frac{1}{\sqrt{4-(x-1)^2}} \, dx = \int \frac{1}{\sqrt{4-4\sin^2 \theta}} d(2\sin \theta + 1) = \int \frac{1}{2\cos \theta} d(2\sin \theta + 1)
= \int \frac{1}{2\cos \theta} 2\cos \theta d\theta = \int 1 \, d\theta = \theta + C
\]

Since \( x-1 = 2\sin \theta \), we have \( \theta = \sin^{-1}(\frac{x-1}{2}) \), and the answer is

\[
\int \frac{1}{\sqrt{3+2x-x^2}} \, dx = \sin^{-1}(\frac{x-1}{2}) + C
\]

Method 2 \( u \) Substitution:

\[
\int \frac{1}{\sqrt{4-(x-1)^2}} \, dx = \int \frac{1}{2\sqrt{1-(\frac{x-1}{2})^2}} \, dx
\]

Since we observe that the integrand is a composite of two functions, the inside function is \( u \). We let \( u = \frac{x-1}{2} \), \( du = \frac{1}{2} \, dx \) and we have:

\[
\int \frac{1}{\sqrt{4-(x-1)^2}} \, dx = \int \frac{1}{2\sqrt{1-(u^2)}} \, dx = \int \frac{1}{\sqrt{1-u^2}} \, du = \sin^{-1} u + C = \sin^{-1}(\frac{x-1}{2}) + C.
\]
Problem 2  [8pt] Evaluate \[ \int \frac{x^2 + 5x + 5}{(x+1)^2(x+2)} \, dx \]

First, check the highest degree in the numerator and denominator separately: degree of numerator is 2, which is smaller than that of denominator (degree is 3). So, everything is good at this step, otherwise we need to use long division to lower the degree of numerator.

Then, we look at the denominator, and it is already factorized into linear terms. (If it is not factorized, then you need to do it by yourself. If in the denominator there is quadratic term \(ax^2 + bx + c\), then you need to check whether it can be factorized into two linear terms by checking \(\Delta = b^2 - 4ac\). If \(\Delta < 0\), then it is not factorizable and your must have term \(\frac{Ax + B}{(ax^2 + bx + c)}\) on the RHS when doing partial fraction. Otherwise if \(\Delta > 0\), \(ax^2 + bx + c\) can be written as \(a(x - x_1)(x - x_2)\), where \(x_1 \neq x_2\), and \(x_1, x_2\) can be either positive or negative, and you have \(A/(x - x_1) + B/(x - x_2)\) on the RHS when doing partial fraction decomposition. If \(\Delta = 0\), then \(ax^2 + bx + c\) can be written as \(a(x - x_1)^2\), and you have \(A/(x - x_1) + B/(x - x_1)^2\) on the RHS.

Solution We do not need to worry about factorization as it is already done. We notice that we only have product of linear terms in the denominator. So, when doing partial fractions, the numerators on the RHS are all constants. (Remember that we need linear numerator \(Ax + B\) only when denominator is unfactorizable quadratic term). Applying partial fractions to the integrand, we have

\[
\frac{x^2 + 5x + 5}{(x+1)^2(x+2)} = \frac{A}{x + 1} + \frac{B}{(x + 1)^2} + \frac{C}{x + 2},
\]

where \(A, B, C\) are constants that need to be determined. By moving denominator on LHS to RHS, we have

\[
x^2 + 5x + 5 = A(x + 1)(x + 2) + B(x + 2) + C(x + 1)^2.
\]

Since the equality above holds for all real number \(x\). By plugging in some wisely choosen numbers, we have solution for \(A, B, C\) immediately, and we do not need to solve linear equations.

Let \(x = -1\), we have \((-1)^2 - 5 + 5 = A \cdot 0 + B \cdot 1 + C \cdot 0\)

\[
B = 1
\]

Let \(x = -2\), we have \(4 - 10 + 5 = A \cdot 0 + B \cdot 0 + C \cdot 1\)

\[
C = -1
\]

Last, we plug in \(x = 0\) (actually any number except \(-1\) and \(-2\) is fine), we have \(0 + 0 + 5 = A \cdot 2 + B \cdot 2 + C \cdot 1\). Since \(B = 1, C = -1\)

\[
A = 2
\]

then

\[
\int \frac{x^2 + 5x + 5}{(x+1)^2(x+2)} \, dx = \int \frac{2}{x + 1} \, d(x + 1) + \int \frac{1}{(x + 1)^2} \, d(x + 1) - \int \frac{1}{x + 2} \, d(x + 2)
\]

\[
= 2 \ln |x + 1| - \frac{1}{x + 1} - \ln |x + 2| + C
\]