Problem 1  [8pt] Evaluate \( \int_0^\infty \frac{1}{e^x + 1} \, dx \)

Solution

Step 1 Find indefinite integral (antiderivative). There are many approaches to find antiderivative:

Method 1 \( u \) substitution:
Since the integrand is composite of two functions, it is natural to try \( u \) substitution, and the inside function is your \( u \). Let \( u = e^x + 1 \), \( du = e^x \, dx = (u - 1) \, dx \). So, \( \int \frac{1}{e^x + 1} \, dx = \int \frac{1}{u(u - 1)} \, du \).

We can use partial fractions to find this integral.

\[
\frac{1}{u(u - 1)} = \frac{A}{u} + \frac{B}{u - 1}
\]

By moving denominator on the LHS to RHS, we have

\[
1 = A(u - 1) + Bu
\]

Let \( u = 0 \), we have \( A = -1 \)

Let \( u = 1 \), we have \( B = 1 \)

Thus,

\[
\int \frac{1}{u(u - 1)} \, du = \int \frac{1}{u - 1} \, du - \int \frac{1}{u} \, du
= \ln |u - 1| - \ln |u| + C
= \ln |e^x| - \ln |e^x + 1| + C
= x - \ln |e^x + 1| + C.
\]

Remark partial fraction needs some computation here and it is possible to have arithmetic mistakes.

Actually in some special cases like this, we do not need much computation and can see the answer immediately. For example, if there are two fractions \( \frac{1}{a} + \frac{1}{b} \), where \( b = a + 1 \). Then \( \frac{1}{ab} = \frac{1}{a} - \frac{1}{b} \). That is, if two factions have numerator 1 and adjacent denominator, then the product of the two fractions is exactly the difference of these two (remember that larger fraction is always substracted by smaller fraction, in this case 1/a is larger than 1/b, so \( \frac{1}{ab} = \frac{1}{a} - \frac{1}{b} \)). In this quiz problem, \( u \) and \( u - 1 \) are adjacent numbers, and \( 1/(u-1) \gg 1/u \) (assuming we have positive numbers), then \( \frac{1}{u(u-1)} = \frac{1}{u-1} - \frac{1}{u} \).

Similarly we have \( \frac{1}{(x+1)(x+2)} = \frac{1}{x+1} - \frac{1}{x+2} \), \( \frac{1}{(x-15)(x-16)} = \frac{1}{x-16} - \frac{1}{x-15} \) etc.

If we do not have adjacent numbers in denominator, as long as the numerator is a constant, we can still see the answer using a generalized argument. For example \( b \) and \( a \) are not adjacent but have some gap \( \gg 1 \). i.e. \( b = a + gap \). Then \( \frac{1}{ab} = \frac{1}{a} \cdot \frac{1}{a + gap} \). That is, the product of the two fractions is exactly the difference of these two divided by the gap of the denominators. e.g. we have \( \frac{1}{(x-1)(x+1)} = \frac{1}{2} \left( \frac{1}{x-1} - \frac{1}{x+1} \right) \), \( \frac{1}{(x+2)(x+5)} = \frac{1}{5} \left( \frac{1}{x+2} - \frac{1}{x+5} \right) \), \( \frac{1}{(x-15)(x-27)} = \frac{1}{12} \left( \frac{1}{x-27} - \frac{1}{x-15} \right) \).

Method 2 \( u \) substitution: We can let \( u = e^x \) and this gives you the same answer. If you haven’t tried, I recommend you do it by yourself.
Method 3 Simplify integrand first:
Remember that in section 7.5, the first strategy is to try some possible simplification of the integrand, which hopefully may make the whole problem much easier.
Actually we can rewrite the integrand in the following way:
\[
\int \frac{1}{e^x + 1} \, dx = \int \frac{1 + e^x - e^x}{e^x + 1} \, dx = \int \left( 1 - \frac{e^x}{e^x + 1} \right) \, dx
\]
Let \( u = e^x \), \( du = e^x \, dx \), we have
\[
\int \left( 1 - \frac{e^x}{e^x + 1} \right) \, dx = x - \int \frac{1}{u + 1} \, du = x - \ln |u + 1| + C = x - \ln |e^x + 1| + C
\]

Step 2 Find improper integral by taking limit
By the definition of the improper integral,
\[
\int_0^\infty \frac{1}{e^x + 1} \, dx = \lim_{b \to \infty} \int_0^b \frac{1}{e^x + 1} \, dx = \lim_{b \to \infty} [(b - \ln |e^b + 1|) - (0 - \ln |e^0 + 1|)] = \lim_{b \to \infty} (b - \ln |e^b + 1|) + \ln 2
\]
Note that the limit \( \lim_{b \to \infty} (b - \ln |e^b + 1|) \) is of the type \( \infty - \infty \), it does NOT mean that the limit does not exist. Actually the limit can exist.
\[
\int_0^\infty \frac{1}{e^x + 1} \, dx = \lim_{b \to \infty} (\ln |e^b| - \ln |e^b + 1|) + \ln 2 = \lim_{b \to \infty} \ln \left| \frac{e^b}{e^b + 1} \right| + \ln 2
\]
by L’Hospital’s Rule
\[
= \ln |\lim_{b \to \infty} \frac{e^b}{e^b + 1}| + \ln 2 = \ln |1| + \ln 2 = \ln 2
\]
Remark The limit in the form of \( \infty - \infty \) may exist, and you need some algebraic trick to verify carefully. For example \( \lim_{x \to \infty} (x - x) \) is of such type, but obviously the limit= \( \lim_{x \to \infty} 0 = 0 \), Same for \( \lim_{x \to \infty} (\sqrt{x^2 + 1} - x) \).
Let me summarize a little bit: \( (+\infty) + (+\infty) = +\infty, \ (-\infty) - (+\infty) = -\infty \). However, \( (+\infty) - (+\infty) \) (same for \( -\infty ) - (-\infty) \) can be \( \infty \) or any finite number, and you need to further check this case.
Problem 2  [7pt] Find length of the curve for $y = e^{\frac{1}{2}x} + e^{-\frac{1}{2}x}$, $0 \leq x \leq 1$
Solution By chain rule,
$$y' = \frac{1}{2}e^{\frac{1}{2}x} - \frac{1}{2}e^{-\frac{1}{2}x}$$

The length of the curve is the following integral:

$$\int_0^1 \sqrt{1 + (y')^2} \, dx = \int_0^1 \sqrt{1 + \left(\frac{1}{2}e^{\frac{1}{2}x} - \frac{1}{2}e^{-\frac{1}{2}x}\right)^2} \, dx$$

$$= \int_0^1 \sqrt{1 + \frac{1}{4}e^x + \frac{1}{4}e^{-x} - \frac{1}{2}} \, dx$$

$$= \int_0^1 \sqrt{\frac{1}{4}e^x + \frac{1}{4}e^{-x} + \frac{1}{2}} \, dx$$

$$= \int_0^1 \sqrt{\left(\frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{2}e^{-\frac{1}{2}x}\right)^2} \, dx$$

$$= \int_0^1 \left(\frac{1}{2}e^{\frac{1}{2}x} + \frac{1}{2}e^{-\frac{1}{2}x}\right) \, dx$$

$$= \int_0^1 \frac{1}{2}e^{\frac{1}{2}x} \, dx + \int_0^1 \frac{1}{2}e^{-\frac{1}{2}x} \, dx$$

For the first integral, let $u = \frac{1}{2}x$. For the second integral, let $u = -\frac{1}{2}x$, then we have

$$\int_0^1 \sqrt{1 + (y')^2} \, dx = (e^{\frac{1}{2}x} - e^{-\frac{1}{2}x})\bigg|_0^1 = e^{\frac{1}{2}} - e^{-\frac{1}{2}}$$