CMO-BIRS SANDPILE WORKSHOP OPEN PROBLEM: MONOMIZATIONS OF POWER IDEALS AND AN INTERVAL DECOMPOSITION OF THE ACYCLIC PARTIAL ORIENTATIONS

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This note grew out of joint work and discussion with Spencer Backman.

Fix $G = (V, E)$, a finite, undirected, connected graph which can have multiple edges but not loops. Designate a distinguished *root vertex* $q \in V$ and set $V^q := V - \{q\}$. All constructions that follow will depend on the pair (G, q), which we call a *pointed graph*. For any $U \subseteq V$ and $u \in U$, define $d_U(u) := \#\{e = \{u, v\} \in E : v \in V - U\}$ and $d_U := \sum_{u\in U} d_U(u)$. Let $\mathbf{k}$ be some field and $R := \mathbf{k}[x_v : v \in V^q]$ the polynomial ring with generators indexed by non-root vertices. For $r \geq -1$ define the following *power ideal* of R:

$$J^r_{(G,q)} := \left\langle \left(\sum_{u\in U} x_u\right)^{d_U+r} : \emptyset \neq U \subseteq V^q \right\rangle .$$

These ideals arise in areas of mathematics as disparate as Schubert calculus and approximation theory. For the complete graph $G = K_n$, the quotient $R/J^{+1}_{(G,q)}$ is canonically isomorphic to the algebra of curvature forms on the flag manifold Fl_{n-1} [27] [28]. For $r = +1, 0, -1$, the inverse systems to $J^r_{(G,q)}$ are the *external*, *central*, and *internal zonotopal algebras* [19] associated to G. These spaces of polynomials are related to the complexity of box splines [13] [14]. The name "zonotopal algebra" comes from the facts that the dimension of the external algebra is equal to the number of lattice points in the graphical zonotope of G, the dimension of the central algebra is equal to the volume of this zonotope, and the dimension of the internal algebra is equal to the number of interior lattice points of this zonotope. Recall that the dimension of the inverse system of an ideal is the same as the dimension of the quotient by that ideal. Stated in terms of the Tutte polynomial $T_G(x, y)$ of G, these claims about the dimension of the zonotopal algebras therefore amount to dim $R/J^{+1}_{(G,q)} = T_G(2, 1)$, dim $R/J^{0}_{(G,q)} = T_G(1, 1)$, and dim $R/J^{-1}_{(G,q)} = T_G(0, 1)$. In fact, from work of Ardila-Postnikov [2] it follows that the Hilbert series of $R/J^r_{(G,q)}$ for $r = +1, 0, -1$ are (generalized) evaluations of the Tutte polynomial of G:

$$\begin{aligned}
\mathrm{Hilb}(R/J^{+1}_{(G,q)}; y) &= y^g \cdot T_G(1 + y, 1/y);\\
\mathrm{Hilb}(R/J^{0}_{(G,q)}; y) &= y^g \cdot T_G(1, 1/y);\\
\mathrm{Hilb}(R/J^{-1}_{(G,q)}; y) &= y^g \cdot T_G(0, 1/y).
\end{aligned}$$

Here $g := \#E - n + 1$ is the cyclomatic genus of G, with $n := \#V$. Ardila-Postnikov establish these formulas by exhibiting a short exact sequence of vector spaces corresponding to deletion-contraction [2, §4.2].

For $\mathcal{J}$ any ideal of a polynomial ring S over $\mathbf{k}$ and $\mathcal{I}$ a monomial ideal of S, we say that $\mathcal{I}$ is a *monomization* of $\mathcal{J}$ if the standard monomials of $\mathcal{I}$ (i.e., the monomials in S not in $\mathcal{I}$) give a $\mathbf{k}$-linear basis of $S/\mathcal{J}$. Of course, a monomization $\mathcal{I}$ of any $\mathcal{J}$ can always be found using Gröbner bases. However, for the $\mathcal{J} = J^r_{(G,q)}$ we are interested in it is computationally expensive to compute a Gröbner basis and all Gröbner bases are "nasty": some heuristic explanations for this nastiness are that the Newton polytopes for the generators of $J^r_{(G,q)}$ are all simplices, which means their leading terms with respect to any term order will be of the form x_u^k; and that while $J^r_{(G,q)}$ is invariant under all automorphisms of G which fix the root q, Gröbner basis theory does not "see" this symmetry. It turns out that the ideals $J^r_{(G,q)}$ have interesting monomizations not arising from Gröbner basis theory. Let $\prec^+, \prec^-$ be total orders on V^q. Define the following monomial ideals of R:

$$\begin{aligned} I^{+1,\prec^+}_{(G,q)} &:= \left\langle x_{\min^{\prec^+}(U)} \cdot \prod_{u\in U} x_u^{d_U(u)} \colon \emptyset \neq U \subseteq V^q \right\rangle; \\ I^{0}_{(G,q)} &:= \left\langle \prod_{u\in U} x_u^{d_U(u)} \colon \emptyset \neq U \subseteq V^q \right\rangle; \\ I^{-1,\prec^-}_{(G,q)} &:= \left\langle x^{-1}_{\min^{\prec^-}(U)} \cdot \prod_{u\in U} x_u^{d_U(u)} \colon \emptyset \neq U \subseteq V^q \right\rangle. \end{aligned}$$

Here $\min^{\prec}(U)$ is the minimum element of U with respect to $\prec$. One extremely nice property of $I^0_{(G,q)}$ is that, like the power ideals, it is invariant under all graph automorphisms that fix the root q. Desjardins [15, Example 7] gives an example which shows that we cannot hope for monomizations of $J^{\pm 1}_{(G,q)}$ to have this property as well and thus the vertex orders $\prec^{\pm}$ are required to break symmetry. Note that the above definition of $I^{-1,\prec^-}_{(G,q)}$ does not always make sense because sometimes these generators will be Laurent monomials rather than honest monomials. But $I^{-1,\prec^-}_{(G,q)}$ is well-defined, for instance, when there is an edge $e = \{q,v\} \in E$ for each $v \in V^q$. At any rate, we have the following theorems about the relationship between the graphical monomial and power ideals.

Theorem 1 ([26, Theorem 3.1])**.** *$I^0_{(G,q)}$ is a monomization of $J^0_{(G,q)}$.*

Theorem 2 ([27, Theorem 2] for $G = K_n$, [15, Theorem 12] for arbitrary G)**.** *For any total order $\prec^+$ on V^q, $I^{+1,\prec^+}_{(G,q)}$ is a monomization of $J^{+1}_{(G,q)}$.*

Theorem 3 ([15, Theorem 22])**.** *If G is* saturated *(meaning that for any $u, v \in V$ there is an edge $e = \{u,v\} \in E$) then for any total order $\prec^-$ on V^q, $I^{-1,\prec^-}_{(G,q)}$ is a monomization of $J^{-1}_{(G,q)}$.*

While it is fair to say that these monomization results are somewhat ad-hoc (in that they are not explained by a more general algebraic theory), the underlying combinatorics is quite interesting and exploring the limits of this approach might point the way to a broader theory of e.g. "Gröbner bases with symmetry." To that end, we remark that the connection between the graphical monomial and power ideals is tighter than monomization: they conjecturally [26, Conjecture 6.10] have the same Betti tables.

The ideal $I^0_{(G,q)}$ is the well-studied (G,q)-*parking function ideal*. Recall that a (G,q)-*parking function* is an element $c = \sum_{v \in V^q} c_v v \in \mathbb{N}V^q$ such that for any $\emptyset \neq U \subseteq V^q$ there is some $u \in U$ with $c_u < d_U(u)$. We call the elements of $\mathbb{N}V^q$ *configurations*. The (G,q)-parking functions are essentially the same as the superstable configurations for the Abelian sandpile model with respect to sink q [25, §2.3]. Equivalently, a configuration c is a (G,q)-parking function if and only if $\mathbf{x}^c := \prod_{v \in V^q} x_v^{c_v}$ is a standard monomial of $I^0_{(G,q)}$. Denote the set of (G,q)-parking functions by $\mathrm{PF}(G,q)$. For $c \in \mathbb{Z}V^q$ define $\deg(c) := \sum_{v \in V^q} c_v$. The fact that $\sum_{c \in \mathrm{PF}(G,q)} y^{\deg(c)} = y^g \cdot T_G(1, 1/y)$, which follows from Theorem 1 together with the aforementioned expression for the Hilbert series of $R/J^0_{(G,q)}$, is a famous result of Merino [23]. By analogy with the central case, we say c is a $(G,q,\prec^+)$-*superparking function* (respectively, $(G,q,\prec^-)$-*subparking function*) if and only if $\mathbf{x}^c$ is a standard monomial of $I^{+1,\prec^+}_{(G,q)}$ (resp., $I^{-1,\prec^-}_{(G,q)}$) and denote the set of these elements of $\mathbb{N}V^q$ by $\mathrm{PF}^+(G,q,\prec^+)$ (resp., $\mathrm{PF}^-(G,q,\prec^-)$). These extended notions of graphical parking functions were introduced by Desjardins [15]; in the case where $G = K_n$ is the complete graph, they also appear in Holtz-Ron [19, Definition 1.5] under the names "external" and "internal" parking functions.

In this expository note we will demonstrate how these extended graphical parking functions can be understood in terms of partial orientations of G. A partial orientation $\mathcal{O}$ of G is a choice for each edge $e \in E$ whether to orient that edge one way, orient it the other way, or to leave it neutral: that is, a partial orientation is an orientation of some subset of the edges of the graph. We say $\mathcal{O}$ is acyclic if it does not contain a directed cycle. Our first goal is to show how a certain interval decomposition of the poset of acyclic partial orientations of G due to Gessel-Sagan [17] is directly related to this story: the interval decomposition can be seen as encoding bijective proofs that $\#\mathrm{PF}^+(G,q,\prec^+) = T_G(2,1)$ and $\#\mathrm{PF}(G,q) = T_G(1,1)$. Next we explain how recent results of Backman-Hopkins [5] suggest that this partial orientation interval decomposition perspective could be pushed further to yield a bijective proof that $\#\mathrm{PF}^-(G,q,\prec^-) = T_G(0,1)$. As can be discerned from the above discussion, this internal case is apparently subtler than the central or external cases. Indeed, the broader aims of this project are: to remedy the unsatisfying situation that there is no construction of a monomization of $J^{-1}_{(G,q)}$ for all G; and to find monomizations for power ideals that interpolate between $J^{+1}_{(G,q)}$, $J^0_{(G,q)}$, and $J^{-1}_{(G,q)}$. For more on this last point, see Remark 8 which reviews recent (Intel STS award-winning) work of Huang [22] on monomizations of power ideals between $J^{+1}_{(G,q)}$ and $J^0_{(G,q)}$.

Let us briefly establish notation for partial orientations. For each edge $e = \{u,v\} \in E$ fix a reference orientation $e^+ := (u,v)$ of that edge and set $e^- := (v,u)$. A *partial orientation* is then just a subset $\mathcal{O}$ of the set $\mathbb{E}(G) := \{e^+, e^- \colon e \in E\}$ of formal symbols such that $e^+ \in \mathcal{O}$ implies $e^- \notin \mathcal{O}$ for all $e \in E$. An edge $e \in E$ with $e^\pm \in \mathcal{O}$ is an *oriented edge* of $\mathcal{O}$ and an edge $e \in E$ such that $e^+, e^- \notin \mathcal{O}$ is a *neutral edge* of $\mathcal{O}$. Denote the set of partial orientations of G by $\mathcal{O}(G)$. The set $\mathcal{O}(G)$ is partially ordered by containment and every interval $[\mathcal{O},\mathcal{O}']$ in this poset is a Boolean lattice. Let $\mathcal{O} \in \mathcal{O}(G)$. For $u,v \in V$, a *directed path* in $\mathcal{O}$ from u to v of length k is a sequence $v_0, e_1^{\delta_1} = (v_0,v_1), v_1, e_2^{\delta_2} = (v_1,v_2), \ldots, e_k^{\delta_k} = (v_{k-1},v_k), v_k$ with $v_0 = u$, $v_k = v$ and $e_i^{\delta_i} \in \mathcal{O}$ and $v_i \in V$ for all i. A *directed cycle* of $\mathcal{O}$ is a directed path in $\mathcal{O}$ from v to v of length $k \geq 1$ for some $v \in V$. We say $\mathcal{O}$ is *acyclic* if it has no directed cycles. Denote the set of acyclic partial orientations of G by $\mathcal{A}(G)$. We say v is *reachable* from u in $\mathcal{O}$ if there is some directed path from u to v of length $k \geq 0$. We say $\mathcal{O}$ is *q-connected* if every $v \in V$ is reachable from q. Denote the set of acyclic, q-connected partial orientations of G by $\mathcal{A}^q(G)$.

In order to relate acyclic partial orientations to the extended parking functions we need a map from partial orientations to configurations. The "indegree minus one" map is often used for this purpose ([7, Theorem 3.1], [24, §10], [4], [21], [20], [1, Theorem 2.1]). We now describe a modification of the "indegree minus one" map that always yields nonnegative coefficients (and a coefficient of 0 at q when the orientation is acyclic). First we need to associate to each partial orientation an ordered partition of the vertices of our graph that records reachability with respect to a given vertex order. Let $\prec$ be a total order on V^q; we will also consider $\prec$ to be a total order on V by designating q to be minimal. Let $\mathcal{O} \in \mathcal{O}(G)$. Define the ordered partition $\Pi^{\prec}(\mathcal{O}) := (\Pi_0^{\prec}(\mathcal{O}),\ldots,\Pi_k^{\prec}(\mathcal{O}))$ of V as follows:

(a) Initialize $i := 0$.
(b) Let v_i be the minimal element with respect to $\prec$ of $V - (\Pi_0^{\prec}(\mathcal{O}) \cup \cdots \cup \Pi_{i-1}^{\prec}(\mathcal{O}))$.
(c) Set $\Pi_i := \{v \in V : v \text{ is reachable from } v_i \text{ in } \mathcal{O}\} - (\Pi_0^{\prec}(\mathcal{O}) \cup \cdots \cup \Pi_{i-1}^{\prec}(\mathcal{O}))$.
(d) If $V \neq \Pi_0^{\prec}(\mathcal{O}) \cup \cdots \cup \Pi_i^{\prec}(\mathcal{O})$, set $i := i+1$ and go to (b). Otherwise terminate.

We use the notation $\#\Pi^{\prec}(\mathcal{O}) := k+1$. Next we define $C^{\prec}(\mathcal{O}) := \sum_{v \in V} c_v v \in \mathbb{Z}V$. Let $v \in V$ with $v \in \Pi_j^{\prec}(\mathcal{O})$ and set

$$c_v := \begin{cases} \widetilde{c}_v & \text{if } v = \min^{\prec}(\Pi_j^{\prec}(\mathcal{O})); \\ \widetilde{c}_v - 1 & \text{otherwise,} \end{cases}$$

where $\widetilde{c}_v := \#\{e^{\pm} = (u,v) \in \mathcal{O} \colon u \in \Pi_j^{\prec}(\mathcal{O})\} + \#\{e^{\pm} = (v,u) \in \mathcal{O} \colon u \in \Pi_i^{\prec}(\mathcal{O}), i < j\}$. Note that in fact $C^{\prec}(\mathcal{O}) \in \mathbb{N}V$ because if a vertex is not the minimal element of its part then it must have at least one incoming oriented edge from a vertex in its part. Furthermore, if $\mathcal{O} \in \mathcal{A}(G)$ then $c_q = 0$ so we can indeed view $C^{\prec}(\mathcal{O}) \in \mathbb{N}V^q$ as a configuration. Moreover, if $\mathcal{O} \in \mathcal{A}^q(G)$ then in fact $C^{\prec}(\mathcal{O}) = \sum_{v \in V^q} (\mathrm{indeg}_{\mathcal{O}}(v) - 1)v$ where $\mathrm{indeg}_{\mathcal{O}}(v) := \#\{e^{\pm} = (u,v) \in \mathcal{O}\}$; in other words, in this case $\mathcal{O} \mapsto C^{\prec}(\mathcal{O})$ really is the "indegree minus one" map. Note that in this case where $\mathcal{O} \in \mathcal{A}^q(G)$ the configuration $C^{\prec}(\mathcal{O})$ does not depend on $\prec$, so we write $C^q(\mathcal{O})$ for this configuration instead.

We also need to review notation for subgraphs, which are closely connected to partial orientations and to the Tutte polynomial. A *subgraph* H of G is a graph $H = (V,S)$ where $S \subseteq E$. In other words, all subgraphs under consideration here include all vertices of the original graph G and some subset of its edges. We thus identify the subgraph $H = (V,S)$ with the subset of edges $S \subseteq E$. Denote the set of subgraphs of G

by $S(G)$. The set $S(G)$ is partially ordered by containment and every interval $[S, S']$ in this poset is a Boolean lattice. Let $S \in S(G)$. For $u, v \in V$, a *path* in S from u to v of length k is a sequence $v_0, e_1 = \{v_0, v_1\}, v_1, e_2 = \{v_1, v_2\}, \ldots, e_k = \{v_{k-1}, v_k\}, v_k$ with $v_0 = u$, $v_k = v$ and $e_i \in S$ and $v_i \in V$ for all i. A *cycle* of S is a path in S from v to v of length $k \geq 1$ for some $v \in V$. We say S is a *forest* if it has no cycles. Denote the set of forests of G by $F(G)$. A *spanning tree* T of G is a forest with $\#T = n - 1$. Denote the set of spanning trees of G by $T(G)$. We say v is *reachable* from u in S if there is some path from u to v of length $k \geq 0$. We associate to each subgraph an ordered partition of the vertices as we did for partial orientations. Let $\prec$ be a total order on V^q and again consider $\prec$ as a total order on V with q minimal. Define the ordered partition $\Pi^{\prec}(S) := (\Pi^{\prec}_0(S), \ldots, \Pi^{\prec}_k(S))$ of V as follows:

(a) Initialize $i := 0$.
(b) Let v_i be the minimal element with respect to $\prec$ of $V - (\Pi^{\prec}_0(S) \cup \cdots \cup \Pi^{\prec}_{i-1}(S))$.
(c) Set $\Pi_i := \{v \in V : v \text{ is reachable from } v_i \text{ in } S\} - (\Pi^{\prec}_0(S) \cup \cdots \cup \Pi^{\prec}_{i-1}(S))$.
(d) If $V \neq \Pi^{\prec}_0(S) \cup \cdots \cup \Pi^{\prec}_i(S)$, set $i := i + 1$ and go to (b). Otherwise terminate.

We want to move between partial orientations and subgraphs, so for $\mathcal{O} \in \mathcal{O}(G)$ we define $S(\mathcal{O}) := \{e \in E : e \text{ is oriented in } \mathcal{O}\} \in S(G)$. Conversely, given $F \in F(G)$ and a total order $\prec$ on V^q, we define $\mathcal{O}^{\prec}(F)$ to be the unique partial orientation with $S(\mathcal{O}^{\prec}(F)) = F$ and $\Pi^{\prec}(\mathcal{O}^{\prec}(F)) = \Pi^{\prec}(F)$; i.e., $\mathcal{O}^{\prec}(F)$ is a "rooted forest" where each connected component is rooted at the minimal element of that component according to $\prec$.

Finally, before we can state the main result (a decomposition of the poset $\mathcal{A}(G)$ into a number of intervals equal to the number of forests of G) we need to review internal and external activities. These activities are fundamental for the Tutte polynomial. An *activity pair* $(\mathrm{in}, \mathrm{ex})$ for G is a pair of maps $\mathrm{in}, \mathrm{ex} \colon T(G) \to S(G)$ such that

- $\mathrm{in}(T) \subseteq T$ and $\mathrm{ex}(T) \subseteq E - T$ for all $T \in T(G)$;
- *(Tutte's expansion)* $T_G(x, y) = \sum_{T \in T(G)} x^{\#\mathrm{in}(T)} y^{\#\mathrm{ex}(T)}$;
- *(Crapo's property)* $S(G) = \bigsqcup_{T \in T(G)} [T - \mathrm{in}(T), T \cup \mathrm{ex}(T)]$ where $\sqcup$ denotes disjoint union.

Of course $\mathrm{in}(T)$ are called the *internally active* edges of T and $\mathrm{ex}(T)$ the *externally active* edges. For $(\mathrm{in}, \mathrm{ex})$ an activity pair, we extend these maps to $\mathrm{in}, \mathrm{ex} \colon S(G) \to S(G)$ by setting $\mathrm{in}(S) := \mathrm{in}(T)$ and $\mathrm{ex}(S) := \mathrm{ex}(T)$ if $S \in [T - \mathrm{in}(T), T \cup \mathrm{ex}(T)]$. The idea of extending activities to arbitrary subgraphs goes back at least to Gordon-Traldi [18]. It is a nontrivial fact that an activity pair exists. We now review the *classical activity pair* $(\mathrm{in}^{<}, \mathrm{ex}^{<})$ which depends on a total order $<$ on E. First we need to discuss cuts and cycles of subgraphs. If $\mathrm{Cy} = v, e_1 = \{v, v_1\}, v_1, \ldots, v_{k-1}, e_k = \{v_{k-1}, v\}, v$ is a cycle of $S \in S(G)$ then we set $E(\mathrm{Cy}) := \{e_1, \ldots, e_k\}$. A *cut* (or cocycle) of S is a partition $\mathrm{Cu} = \{U, V - U\}$ of V with $E(\mathrm{Cu}) := \{e = \{u, v\} \in E : u \in U, v \in V - U\} \subseteq S$. Now fix some $T \in T(G)$. Let $e \in T$. Then $e \in \mathrm{in}^{<}(T)$ if and only if $e = \min^{<}(E(\mathrm{Cu}))$ where Cu is the unique cut of $(E - T) \cup \{e\}$. Now let $e \in E - T$. Then $e \in \mathrm{ex}^{<}(T)$ if and only if $e = \min^{<}(E(\mathrm{Cy}))$ where Cy is the unique cycle of $T \cup \{e\}$. The fact that the pair $(\mathrm{in}^{<}, \mathrm{ex}^{<})$ satisfies Tutte's expansion is a result of Tutte [29] and the fact that it satisfies Crapo's property is a result of Crapo [12]. Another example of an activity pair is due to Bernardi [8]: this activity pair depends on a combinatorial map (embedding of the graph in a surface) rather than a total edge order. In his PhD thesis work, Courtiel [11] offers a general framework for understanding activity pairs. One notion of activity that is highly relevant to our present concern is the *depth-first search (DFS) external activity* of Gessel and Sagan [17]. DFS external activity is closely related to the inversion number of ordered trees. However, it is unclear, at least to me, what notion of internal activity pairs with DFS external activity.

Theorem 4. *Fix a pointed graph (G, q) and a total order $\prec^+$ on V^q. Then there exist an activity pair* $(\mathrm{in}, \mathrm{ex})$ *for G and maps $\mathcal{O}_{\min}, \mathcal{O}_{\max} \colon F(G) \to \mathcal{O}(G)$ satisfying*

(1) $\mathcal{O}_{\min}(F) \leq \mathcal{O}_{\max}(F)$ *for all $F \in F(G)$;*
(2) $\Pi^{\prec^+}(\mathcal{O}) = \Pi^{\prec^+}(F)$ *for all $\mathcal{O} \in [\mathcal{O}_{\min}(F), \mathcal{O}_{\max}(F)]$ and $F \in F(G)$;*
(3) $S(\mathcal{O}_{\min}(F)) = F$ *for all $F \in F(G)$;*
(4) $S(\mathcal{O}_{\max}(F)) = E - \mathrm{ex}(F)$ *for all $F \in F(G)$;*
(5) $\mathcal{A}(G) = \bigsqcup_{F \in F(G)} [\mathcal{O}_{\min}(F), \mathcal{O}_{\max}(F)]$*;*
(6) $\mathcal{A}^q(G) = \bigsqcup_{T \in T(G)} [\mathcal{O}_{\min}(T), \mathcal{O}_{\max}(T)]$*;*
(7) $F \mapsto C^{\prec^+}(\mathcal{O}_{\max}(F))$ *bijects between $F(G)$ and* $\mathrm{PF}^+(G, q, \prec^+)$*;*
(8) $T \mapsto C^q(\mathcal{O}_{\max}(T))$ *bijects between $T(G)$ and* $\mathrm{PF}(G, q)$*.*

The existence of an interval decomposition satisfying properties (1)-(6) of Theorem 4 was established by Gessel and Sagan [17, §3]. They apparently did not know about graphical parking functions (which makes sense because that terminology originated in the later work of Postnikov-Shapiro [26]). Gessel and Sagan used DFS and DFS external activity in their interval decomposition. Instead of DFS external activity we will use the classical activity pair and instead of DFS we will employ an extended version of the Cori-Le Borgne variant [10] of Dhar's burning algorithm [16]. The Cori-Le Borgne procedure should just be seen as a different way of exploring the graph than DFS, one that depends on a total order of the edges. There are many possible ways to explore a graph (depth-first search, bread-first search, Cori-Le Borgne, etc.) and each of these leads to some specialization of the "generic Dhar's algorithm." Indeed, any specialization of the generic Dhar's algorithm could be used to establish Theorem 4. Each specialization corresponds to a choice of external activity and relates the degree of a parking function (superparking function) to a different statistic of spanning trees (forests). Although Theorem 4 seems to only concern external activity, internal activity plays a central role in Conjecture 11, the main open problem of this exposition. Before we prove Theorem 4 let us state some of its immediate consequences.

Corollary 5. *With the set-up of Theorem 4:*

- $\sum_{\mathcal{O} \in \mathcal{A}(G)} x^{\#\Pi^{\prec^+}(\mathcal{O})} y^{\#\mathcal{O}} = x\, y^{n-1}\, (1+y)^g \cdot T_G(1 + xy^{-1}(1+y), (1+y)^{-1})$*;*
- $\sum_{\mathcal{O} \in \mathcal{A}^q(G)} y^{\#\mathcal{O}} = y^{n-1}\, (1+y)^g \cdot T_G(1, (1+y)^{-1})$*;*
- $\mathrm{Hilb}(R/I^{+1,\prec^+}_{(G,q)}) = \sum_{c \in \mathrm{PF}^+(G,q,\prec^+)} y^{\deg(c)} = y^g \cdot T_G(1+y, 1/y)$*;*
- $\mathrm{Hilb}(R/I^{0}_{(G,q)}) = \sum_{c \in \mathrm{PF}(G,q)} y^{\deg(c)} = y^g \cdot T_G(1, 1/y)$*;*
- $\{C^{\prec^+}(\mathcal{O}) \colon \mathcal{O} \in \mathcal{A}(G)\} = \mathrm{PF}^+(G, q, \prec^+)$*;*
- $\{C^q(\mathcal{O}) \colon \mathcal{O} \in \mathcal{A}^q(G)\} = \mathrm{PF}(G, q)$*.*

Proof. The generating functions follow from formal manipulations invoking the properties (1)-(8) and recalling that every interval in $\mathcal{O}(G)$ or $S(G)$ is a Boolean lattice. For the first bullet point, we have

$$\begin{aligned}
\sum_{\mathcal{O}\in\mathcal{A}(G)} x^{\#\Pi^{\prec^+}(\mathcal{O})} y^{\#\mathcal{O}} &= \sum_{F\in F(G)} \sum_{\mathcal{O}\in[\mathcal{O}_{\min}(F),\mathcal{O}_{\max}(F)]} x^{\#\Pi^{\prec^+}(\mathcal{O})} y^{\#\mathcal{O}} \\
&= \sum_{F\in F(G)} x^{\#\Pi^{\prec+}(F)} y^{\#\mathcal{O}_{\min}(F)} \sum_{\mathcal{O}\in[\mathcal{O}_{\min}(F),\mathcal{O}_{\max}(F)]} y^{\#(\mathcal{O}-\mathcal{O}_{\min}(F))} \\
&= \sum_{F\in F(G)} x^{n-\#F} y^{\#F} (1+y)^{\#(E-\mathrm{ex}(F)-F)} \\
&= \sum_{T\in T(G)} \sum_{F\in[T-\mathrm{in}(T),T]} x^{n-\#F} y^{\#F} (1+y)^{\#(E-F)-\#\mathrm{ex}(F)} \\
&= \sum_{T\in T(G)} x^{n-\#T} y^{\#T} (1+y)^{\#(E-T)-\#\mathrm{ex}(T)} \sum_{F\in[T-\mathrm{in}(T),T]} (xy^{-1}(1+y))^{\#(T-F)} \\
&= x\, y^{n-1}\, (1+y)^g \sum_{T\in T(G)} (1+y)^{-\#\mathrm{ex}(T)} (1+xy^{-1}(1+y))^{\#\mathrm{in}(T)} \\
&= x\, y^{n-1}\, (1+y)^g \cdot T_G(1+xy^{-1}(1+y), (1+y)^{-1}).
\end{aligned}$$

The second bullet point is similar. For the third bullet point, we have

$$\begin{aligned}
\sum_{c\in\mathrm{PF}(G,q)} y^{\deg(c)} &= \sum_{F\in F(G)} y^{\deg(C^{\prec^+}(\mathcal{O}_{\max}(F)))} \\
&= \sum_{F\in F(G)} y^{\#\mathcal{O}_{\max}(F)+\#\Pi^{\prec^+}(\mathcal{O}_{\max}(F))-n} \\
&= \sum_{F\in F(G)} y^{\#(E-\mathrm{ex}(F))+\#\Pi^{\prec^+}(F)-n} \\
&= \sum_{T\in T(G)} \sum_{F\in[T-\mathrm{in}(T),T]} y^{\#E-\#\mathrm{ex}(F)-\#F} \\
&= \sum_{T\in T(G)} y^{\#(E-T)-\#\mathrm{ex}(T)} \sum_{F\in[T-\mathrm{in}(T),T]} y^{\#(T-F)} \\
&= y^g \sum_{T\in T(G)} y^{-\#\mathrm{ex}(T)} (1+y)^{\#\mathrm{in}(T)} \\
&= y^g \cdot T_G(1+y, 1/y).
\end{aligned}$$

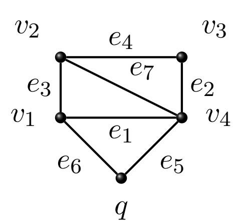
The fourth bullet point is similar. For the fifth bullet point, if $c \in \mathrm{PF}^+(G, q, \prec^+)$ then by (7) and (5) there is $\mathcal{O} \in \mathcal{A}(G)$ such that $c = C^{\prec^+}(\mathcal{O})$. Conversely, if $\mathcal{O} \in \mathcal{A}(G)$ then by (5) there is $F \in F(G)$ for which $\mathcal{O} \in [\mathcal{O}_{\min}(F), \mathcal{O}_{\max}(F)]$, and (2) implies that $C^{\prec^+}(\mathcal{O}_{\min}(F)) \leq C^{\prec^+}(\mathcal{O}) \leq C^{\prec^+}(\mathcal{O}_{\max}(F))$, but (3) implies $C^{\prec^+}(\mathcal{O}_{\min}(F)) = 0$ and (7) implies $C^{\prec^+}(\mathcal{O}_{\max}(F)) \in \mathrm{PF}^+(G, q, \prec^+)$, so $C^{\prec^+}(\mathcal{O}) \in \mathrm{PF}^+(G, q, \prec^+)$. The sixth bullet point is similar. $\square$

Sketch of proof of Theorem 4. As mentioned above, we will take our activity pair to be $(\mathrm{in}^<, \mathrm{ex}^<)$ for $<$ an arbitrary edge order. From properties (2) and (3) it is clear that we must set $\mathcal{O}_{\min}(F) := \mathcal{O}^{\prec^+}(F)$ for all $F \in F(G)$. Thus the problem is to define $\mathcal{O}_{\max}(F)$. It is to this end that we apply an extension of the Cori-Le Borgne variant of Dhar's burning algorithm. We need one more bit of notation for this algorithm: for $U, U' \subseteq V$ we set $\mathbb{E}(U, U') := \{e^{\pm} = (u, u') \colon u \in U, u' \in U', e \in E\}$. Now fix a forest $F \in F(G)$. The algorithm proceeds as follows:

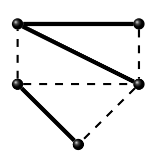
(I) Initialize $i := 0$ and $B_i := \{q\}$ and $\mathcal{O}_i := \emptyset$.
(II) Set $\mathbb{E}_i := \mathbb{E}(B_i, V - B_i) - \{e^{\pm} \colon e \in S(\mathcal{O}_i)\}$.
(III) If $\mathbb{E}_i \neq \emptyset$, then:
  (i) Set $e_i^{\delta_i} := \max^<(\mathbb{E}_i)$.[1]
  (ii) Set $\mathcal{O}_{i+1} := \mathcal{O}_i \cup \{e_i^{\delta_i}\}$.
  (iii) If $e_i = \{u, v\} \in F$ with $u \in B_i$, set $B_{i+1} := B_i \cup \{v\}$.
  (iv) Otherwise (i.e. if $e_i \notin F$), set $B_{i+1} := B_i$.
(IV) If $\mathbb{E}_i = \emptyset$, then:
  (i) Set $\mathcal{O}_{i+1} := \mathcal{O}_i - \mathbb{E}(B_i, V - B_i) \cup \mathbb{E}(V - B_i, B_i)$ (in other words, reverse the orientation of all edges in $E(\{B_i, V - B_i\})$).
  (ii) Set $B_{i+1} := B_i \cup \min^{\prec^+}(V - B_i)$.
(V) Set $i := i + 1$. If $B_i = V$ then terminate. Otherwise, go to (II).

The elements of B_i are the *burning vertices* at step i and the elements of $V - B_i$ are the vertices that are not yet burning at step i. We define $\mathcal{O}_{\max}(F) := \mathcal{O}_i$ where i is the value it attains when the algorithm terminates (which will be $i = \#(E - \mathrm{ex}(F)) + \#\Pi^{\prec^+}(F) - 1$). This algorithm may be a bit much to parse all at once, so let us give an example run to clarify the situation.

Example 6. Let (G, q) be as below:



Let $\prec^+$ be given by $q \prec^+ v_1 \prec^+ v_2 \prec^+ v_3 \prec^+ v_4$ and $<$ be given by $e_1 < e_2 < \cdots < e_7$. Take F to be as below (where solid edges belong to F and dashed edges do not):



[1]This choice, the choice of which edge to "burn along", is the "specialization" of the generic Dhar's burning algorithm. The Cori-Le Borgne procedure chooses according to a total edge order. DFS would select an edge according to a different procedure that depends on the order in which vertices are added to the set of burning vertices.

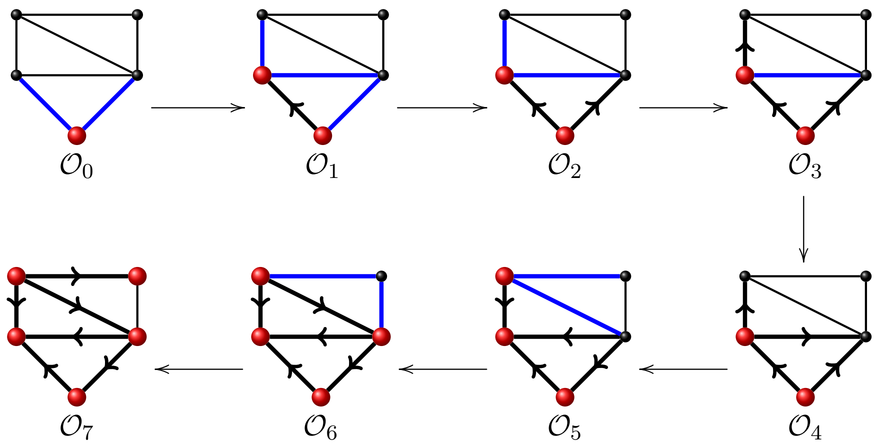
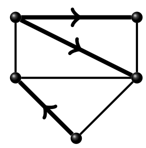


FIGURE 1. Example 6: a run of the extended Cori-Le Borgne algorithm. The vertices in B_i are red and the edges of $\mathbb{E}_i$ are blue.

Then $\mathcal{O}_{\min}(F) = \mathcal{O}^{\prec^+}(F)$ is



and $\mathcal{O}_{\max}(F)$ is

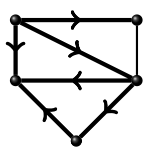


Figure 1 depicts the steps of the Cori-Le Borgne algorithm that compute $\mathcal{O}_{\max}(F)$. Note $\Pi^{\prec^+}(F) = \Pi^{\prec^+}(\mathcal{O}_{\min}(F)) = \Pi^{\prec^+}(\mathcal{O}_{\max}(F)) = (\{q, v_1\}, \{v_2, v_3, v_4\})$. Also note that $\mathrm{ex}(F) = \{e_2\}$ and $S(\mathcal{O}_{\max}(F)) = E - \{e_2\}$. Also note $C^{\prec^+}(\mathcal{O}_{\max}(F)) = v_2 + 2v_4$ and indeed $v_2 + 2v_4 \in \mathrm{PF}^+(G, q, \prec^+)$. $\triangle$

Verifying that properties (1)-(4) hold is routine. Let us now verify (5). It is clear from construction that $\mathcal{O}_{\max}(F)$ is acyclic for all $F \in F(G)$. And observe that $\mathcal{O} \leq \mathcal{O}'$ for some acyclic partial orientation $\mathcal{O}'$ implies $\mathcal{O}$ is acyclic as well. Thus we certainly have the inclusion $\cup_{F\in F(G)}[\mathcal{O}_{\min}(F), \mathcal{O}_{\max}(F)] \subseteq \mathcal{A}(G)$. Next let us show the reverse inclusion: for any $\mathcal{O} \in \mathcal{A}(G)$ there is $F \in F(G)$ with $\mathcal{O} \in [\mathcal{O}_{\min}(F), \mathcal{O}_{\max}(F)]$. Basically we "run the same algorithm" again; specifically, we do the following:

(I') Initialize $i := 0$ and $B_i := \{q\}$ and $\mathcal{O}_i := \emptyset$ and $F_i := \emptyset$.

(II') Set $\mathbb{E}_i := (\mathbb{E}(B_i, V - B_i) - \{e^{\pm} \colon e \in S(\mathcal{O}_i)\}) \cap \mathcal{O}$.

(III') If $\mathbb{E}_i \neq \emptyset$, then:

  (i) Set $e_i^{\delta_i} := \max^{<}(\mathbb{E}_i)$.

(ii) Set $\mathcal{O}_{i+1} := \mathcal{O}_i \cup \{e_i^{\delta_i}\}$.
(iii) If $e_i^{\delta_i} = (u,v)$ and there is no other $e^{\pm} = (w,v) \in \mathbb{E}_i$ for any $w \in B_i$, set $B_{i+1} := B_i \cup \{v\}$ and $F_{i+1} := F_i \cup \{e\}$.
(iv) Otherwise, set $B_{i+1} := B_i$ and $F_{i+1} := F_i$.

(IV') If $\mathbb{E}_i = \emptyset$, then:
(i) Set $\mathcal{O}_{i+1} := \mathcal{O}_i - \mathbb{E}(B_i, V-B_i) \cup \mathbb{E}(V-B_i, B_i)$.
(ii) Set $B_{i+1} := B_i \cup \min^{\prec^+}(V-B_i)$ and $F_{i+1} := F_i$.

(V') Set $i := i+1$. If $B_i = V$ then terminate. Otherwise, go to (II').

Set $F := F_i$ where i is the value it attains when this algorithm terminates. Then we can check $\mathcal{O} \in [\mathcal{O}_{\min}(F), \mathcal{O}_{\max}(F)]$. At this point we could finish the proof that (5) holds by appealing to results of Backman [3] or Backman-Hopkins [5] who show via deletion-contraction that the number of acyclic partial orientations is $2^g \cdot T_G(3, \frac{1}{2})$. But it is also not hard to show directly that these intervals are disjoint. Let $F, F' \in F(G)$ be two different forests. We have just seen that we can recover F from $\mathcal{O}_{\max}(F)$, so $\mathcal{O}_{\max}(F) \neq \mathcal{O}_{\max}(F')$. This means in particular that $\mathcal{O}_i \neq \mathcal{O}'_i$ for some i (where the non-prime variables are from the algorithm applied to F and the prime ones are from the algorithm applied to F'). But by the deterministic nature of the algorithm this means in particular that there is some i with $B_i \neq B'_i$. Choose a minimal such i and suppose without loss of generality that $B_i = B'_i \cup \{v\}$. Note that for some $j > i$ we have $v \in B'_j$. Choose a minimal such j. Either v was added to B'_j as a result of line (III)(iii) of the algorithm or as a result of line (IV)(ii). Suppose it was added as a result of line (III)(iii). Then $(e'_j)^{\delta'_j} \in \mathcal{O}_{\min}(F')$ but $(e'_j)^{\delta'_j} \notin \mathcal{O}_{\max}(F)$ so we have $[\mathcal{O}_{\min}(F), \mathcal{O}_{\max}(F)] \cap [\mathcal{O}_{\min}(F'), \mathcal{O}_{\max}(F')] = \emptyset$. Next suppose v was added as a result of (IV)(ii). Then $e_i^{-\delta_i} \in \mathcal{O}_{\max}(F')$ and so $e_i^{\delta_i} \notin \mathcal{O}_{\max}(F')$; but $e_i^{\delta_i} \in \mathcal{O}_{\min}(F)$, so again we conclude $[\mathcal{O}_{\min}(F), \mathcal{O}_{\max}(F)] \cap [\mathcal{O}_{\min}(F'), \mathcal{O}_{\max}(F')] = \emptyset$. Thus we have shown (5) holds. The proof of (6) is very similar: this time we use the fact that $\mathcal{O}' \leq \mathcal{O}$ with $\mathcal{O}'$ q-connected implies that $\mathcal{O}$ is also q-connected.

Finally let us verify (7). First let us show that for any $c \in \mathrm{PF}^+(G, q, \prec^+)$ there is some forest $F \in F(G)$ with $c = C^{\prec^+}(\mathcal{O}_{\max}(F))$. Again, to do this we just "run the same algorithm." Specifically, suppose $c = \sum_{v \in V^q} c_v v$; then we do the following:

(I") Initialize $i := 0$ and $B_i := \{q\}$ and $\mathcal{O}_i := \emptyset$ and $F_i := \emptyset$.

(II") Set $\mathbb{E}_i := \mathbb{E}(B_i, V-B_i) - \{e^{\pm} \colon e \in S(\mathcal{O}_i)\}$.

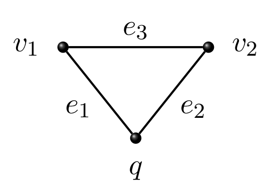
(III") If $\mathbb{E}_i \neq \emptyset$, then:
(i) Set $e_i^{\delta_i} := \max^{<}(\mathbb{E}_i)$.
(ii) Set $\mathcal{O}_{i+1} := \mathcal{O}_i \cup \{e_i^{\delta_i}\}$.
(iii) If $\#\{e = \{w,v\} \colon w \in B_i, e \in S(\mathcal{O}_{i+1})\} = c_v + 1$ where $e_i = \{u,v\}$ with $u \in B_i$ and $v \in V - B_i$, set $B_{i+1} := B_i \cup \{v\}$ and $F_{i+1} := F_i \cup \{e\}$.
(iv) Otherwise, set $B_{i+1} := B_i$ and $F_{i+1} := F_i$.

(IV") If $\mathbb{E}_i = \emptyset$, then:
(i) Set $\mathcal{O}_{i+1} := \mathcal{O}_i - \mathbb{E}(B_i, V-B_i) \cup \mathbb{E}(V-B_i, B_i)$.
(ii) Set $B_{i+1} := B_i \cup \min^{\prec^+}(V-B_i)$ and $F_{i+1} := F_i$.

(V") Set $i := i+1$. If $B_i = V$ then terminate. Otherwise, go to (II").

Set $F := F_i$ where i is the value it attains when this algorithm terminates. We claim that $c = C^{\prec^+}(\mathcal{O}_{\max}(F))$. Define the coefficients c^i_v by $C^{\prec^+}(\mathcal{O}_i) = \sum_{v\in V^q} c^i_v v$. It is not hard to see that if $v \in B_i$ and $j \geq i$ then $c^i_v = c^j_v$. Also if $B_i = B_{i-1} \cup \{v\}$ and v was added to B_i as a result of (III")(iii) then $c^i_v = c_v$. So we need only show that if v was added as a result of (IV")(ii) then we still have $c^i_v = c_v$. Suppose not. Then if we set $U := V - B_{i-1}$ we will have that $x_v \cdot \prod_{u\in U} x_u^{d_U(u)}$ divides $\mathbf{x}^c$, contradicting the assumption $c \in \mathrm{PF}^+(G, q, \prec^+)$. So indeed $c = C^{\prec^+}(\mathcal{O}_{\max}(F))$. At this point we could finish the proof that (7) holds by appealing to the fact that $\#\mathrm{PF}^+(G, q, \prec^+) = T_G(2, 1)$ which follows from Theorem 2. But it is not hard to just finish the proof without this fact either. Let us show $C^{\prec^+}(\mathcal{O}_{\max}(F)) \neq C^{\prec^+}(\mathcal{O}_{\max}(F'))$ for $F \neq F' \in F(G)$. As in the proof of the disjointness of the intervals, let i be minimal with with $B_i \neq B'_i$ and suppose without loss of generality that $B_i = B'_i \cup \{v\}$. Then for some minimal $j > i$ we have $v \in B'_j$ and either v is added to B'_j as a result of line (III)(iii) or of line (IV)(ii) but either way $(c')^j_v > c^i_v$ which shows $C^{\prec^+}(\mathcal{O}_{\max}(F)) \neq C^{\prec}(\mathcal{O}_{\max}(F'))$. Finally let us show $C^{\prec^+}(\mathcal{O}) \in \mathrm{PF}^+(G, q, \prec^+)$ for all $\mathcal{O} \in \mathcal{A}(G)$. So let $\mathcal{O} \in \mathcal{A}(G)$. Suppose to the contrary that $c := C^{\prec^+}(\mathcal{O}) \notin \mathrm{PF}^+(G, q, \prec^+)$. Then $\mathbf{x}^U := x_{\min^{\prec^+}(U)} \cdot \prod_{u\in U} x_u$ divides $\mathbf{x}^c$ for some $\emptyset \neq U \subseteq V^q$. Let j be minimal such that $\Pi^{\prec^+}_j(\mathcal{O}) \cap U \neq \emptyset$. First suppose that $\min^{\prec^+}(\Pi^{\prec^+}_j(\mathcal{O})) \notin U$. Then in order for $\mathbf{x}^U$ to divide $\mathbf{x}^c$ we need that for each $v \in \Pi^{\prec^+}_j(\mathcal{O}) \cap U$ there is $e^\pm = (u, v) \in \mathcal{O}$ with $u \in \Pi^{\prec^+}_j(\mathcal{O}) \cap U$. But this means $\mathcal{O}$ contains a directed cycle involving vertices in $\Pi^{\prec^+}_j(\mathcal{O}) \cap U$, contradicting that $\mathcal{O} \in \mathcal{A}(G)$. Now suppose that $\min^{\prec^+}(\Pi^{\prec^+}_j(\mathcal{O})) \in U$. Then in fact $\min^{\prec^+}(\Pi^{\prec^+}_j(\mathcal{O})) = \min^{\prec^+}(U)$. But then in order for $\mathbf{x}^U$ to divide $\mathbf{x}^c$, because of the "extra factor" of $x_{\min^{\prec^+}(U)}$ in $\mathbf{x}^U$ we still need that for each $v \in \Pi^{\prec^+}_j(\mathcal{O}) \cap U$ there is some $e^\pm = (u, v) \in \mathcal{O}$ with $u \in \Pi^{\prec^+}_j(\mathcal{O}) \cap U$, which again forces a directed cycle in $\mathcal{O}$. So (7) holds. The proof of (8) is similar. Actually, (8) just asserts the correctness of the Cori-Le Borgne algorithm which is proved in [10]. See also [6, §5.2] for another description of the Cori-Le Borgne algorithm. $\square$

Example 7. Let (G, q) be the pointed triangle graph K_3 as below:



Take $v_1 \prec^+ v_2$ and $e_1 < e_2 < e_3$. Then Figure 2 records the interval decomposition of $\mathcal{A}(G)$ resulting from the proof of Theorem 4. $\triangle$

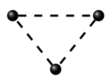
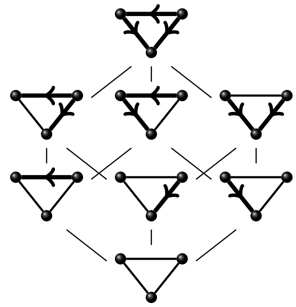
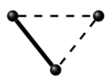
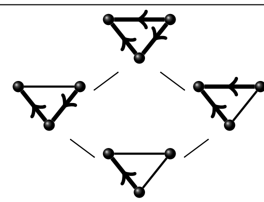
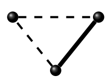
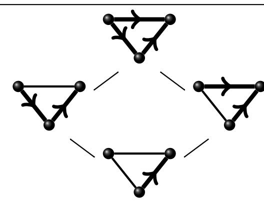
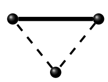
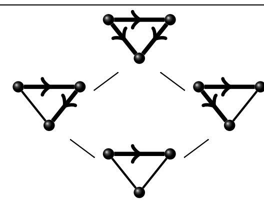
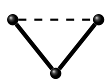
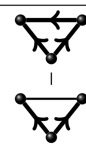
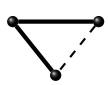
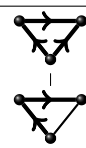
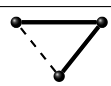
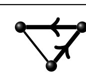
$F \in F(G)$	$\mathrm{in}^{<}(F)$	$\mathrm{ex}^{<}(F)$	$[\mathcal{O}_{\min}(F), \mathcal{O}_{\max}(F)]$	$C^{\prec^+}(\mathcal{O}_{\max}(F))$
	$\{e_1, e_2\}$	$\emptyset$		$v_1 + 2v_2$
	$\{e_1, e_2\}$	$\emptyset$		$2v_2$
	$\{e_1, e_2\}$	$\emptyset$		$2v_1$
	$\{e_1\}$	$\emptyset$		$v_1 + v_2$
	$\{e_1, e_2\}$	$\emptyset$		v_1
	$\{e_1\}$	$\emptyset$		v_2
	$\emptyset$	$\{e_1\}$		0

FIGURE 2. Example 7: the interval decomposition of $\mathcal{A}(G)$ for $G = K_3$.

Remark 8. Let Δ be a simplicial complex on V^q. Brice Huang [22] defines the ideals

$$J^{+1,\Delta}_{(G,q)} := \left\langle \left(\sum_{u\in U} x_u\right)^{d_U+\delta(U\in\Delta)} : \emptyset \neq U \subseteq V^q \right\rangle$$

$$I^{+1,\Delta,\prec^+}_{(G,q)} := \left\langle x^{\delta(U\in\Delta)}_{\min^{\prec^+}(U)} \cdot \prod_{u\in U} x_u^{d_U(u)} : \emptyset \neq U \subseteq V^q \right\rangle$$

where $\delta(U\in\Delta)$ is 1 if $U \in \Delta$ and 0 if $U \notin \Delta$. He shows [22, Theorem 1.4] that $I^{+1,\Delta,\prec^+}_{(G,q)}$ is a monomization of $J^{+1,\Delta}_{(G,q)}$ for any total order $\prec^+$ on V^q and any Δ. Of course this result reduces to Theorem 1 when $\Delta = \emptyset$ and to Theorem 2 when $\Delta = \mathcal{P}(V^q)$ and in a sense interpolates between those two results. Huang shows moreover that the dimension of $R/J^{+1,\Delta}_{(G,q)}$ is the number of Δ-forests of G, where we say $F \in F(G)$ is a Δ-forest if $V^q - \Pi_0^{\prec^+}(F) \in \Delta$. It should be the case that $F \mapsto C^{\prec^+}(\mathcal{O}_{\max}(F))$ (where $\mathcal{O}_{\max}$ is as in our proof of Theorem 4) bijects between Δ-forests and "Δ-superparking functions" (i.e., $c \in \mathbb{N}V^q$ such that $\mathbf{x}^c$ is a standard monomial of $I^{+1,\Delta,\prec^+}_{(G,q)}$). There is an obvious definition of acyclic, (Δ, q)-connected partial orientations and they should fit perfectly into this story. Indeed, Huang's approach is quite similar to what is described in the proof of Theorem 4; but he extends a spanning tree bijection due to Chebikin-Pylyavskyy [9] rather than the Cori-Le Borgne algorithm. △

Remark 9. Let $A = (a_{e^\pm}) \in \mathbb{R}^{\mathbb{E}(G)}_{>0}$ be a *parameter list*, a list of positive real parameters $a_{e^+}, a_{e^-} \in \mathbb{R}_{>0}$ for each $e \in E(G)$. Let $W \simeq \mathbb{R}^{V(G)}$ be a real vector space with basis x_v for $v \in V$. The *bigraphical arrangement* $\Sigma_G(A) \subseteq W$ with respect to A is

$$\Sigma_G(A) := \{H_{e^+}, H_{e^-} : e \in E(G) \text{ with } e \text{ not a loop}\}$$

where for $e \in E(G)$ with $e^\pm = (u, v)$ we define $H_{e^\pm} := x_v - x_u = a_{e^\pm}$. The hyperplanes H_{e^+} and H_{e^-} cut out a "sandwich" in space for each $e \in E(G)$, so that for any region R of $\Sigma_G(A)$ exactly one of the following holds:

(a) R is in the half-space of $W \setminus H_{e^+}$ opposite from H_{e^-};
(b) R is in the half-space of $W \setminus H_{e^-}$ opposite from H_{e^+};
(c) R is between H_{e^+} and H_{e^-}.

Thus there is a natural map $R \mapsto \mathcal{O}_R$ that associates to any region R of the bigraphical arrangement a partial orientation $\mathcal{O}_R$ of $(G, \mathcal{O}_{\text{ref}})$ whereby $e \in E(G)$ is oriented as e^+ in case (a), it is oriented as e^- in case (b), and it is left neutral in case (c). A partial orientation $\mathcal{O}$ is called *A-admissible* if $\mathcal{O} = \mathcal{O}_R$ for some region R of $\Sigma_G(A)$. Clearly if $\mathcal{O}$ is A-admissible then $\mathcal{O} \in \mathcal{A}(G)$. The bigraphical arrangement was introduced by Hopkins-Perkinson [20]. Their main result [20, Corollary 2.8] is that for any parameter list $A \in \mathbb{R}^{\mathbb{E}(G)}_{>0}$ we have

$$\left\{\sum_{v\in V} \mathrm{indeg}_{\mathcal{O}}(v)v : \mathcal{O} \text{ is } A\text{-admissible}\right\} = \left\{\sum_{v\in V} \mathrm{indeg}_{\mathcal{O}}(v)v : \mathcal{O} \in \mathcal{A}(G)\right\}.$$

In fact, using the techniques in [20] as well as Theorem 4, one can show

$$\left\{C^{\prec^+}(\mathcal{O})\colon \mathcal{O} \text{ is } A\text{-admissible}\right\} = \left\{C^{\prec^+}(\mathcal{O})\colon \mathcal{O} \in \mathcal{A}(G)\right\}$$

for any parameter list $A \in \mathbb{R}_{>0}^{\mathbb{E}(G)}$, choice of root $q \in V$, and total order $\prec^+$ on V^q. $\triangle$

Remark 10. Observe that $\mathcal{A}(G)$ is a pure simplicial complex of dimension $\#E - 1$. It would be extremely interesting to realize the map $\mathcal{O} \mapsto C^{\prec^+}(\mathcal{O})$ as an algebraic map from the Stanley-Reisner ring of $\mathcal{A}(G)$ to $R/J^{+1}_{(G,q)}$. $\triangle$

So far what we have covered is mostly a summary of known results. We are now almost ready to state our main conjecture which asserts that there is a class of partial orientations related to the subparking functions in the same way that Theorem 4 shows $\mathcal{A}(G)$ is related to the superparking functions and $\mathcal{A}^q(G)$ is related to the parking functions. First we need to review a few notions from [5]. A *q-rooted spanning tree T* is a spanning tree $T \in T(G)$ together with

- a partial order $\preccurlyeq_T$ on V^q whereby $u \preccurlyeq_T v$ if and only if the unique path from v to q in T passes through u;
- a partial order $\leq_T$ on T whereby $e_1 \leq e_2$ if and only if $e_1 = \{u, v\}$ with $u \preccurlyeq_T v$ and such that the unique path from v to q in T passes through e_2.

Note that the ideal $I^{-1,\prec^-}_{(G,q)}$ is well-defined precisely when there is a q-rooted spanning tree T of G such that $\prec^-$ is a linear extension of $\preccurlyeq_T$. An *ordered, q-rooted spanning tree* T is a q-rooted spanning tree T together with a total order $<_T$ on T that is a linear extension of $\leq_T$. A *potential cut* of a partial orientation $\mathcal{O} \in \mathcal{O}(G)$ is a cut $\mathrm{Cu} = \{U, V - U\}$ such that there are not $e_1, e_2 \in E(\mathrm{Cu})$ with $e_1^{\delta_1} = (u_1, v_1) \in \mathcal{O}$ and $e_2^{\delta_2} = (v_2, u_2) \in \mathcal{O}$ where $u_1, u_2 \in U$ and $v_1, v_2 \in V - U$. In other words, a potential cut of $\mathcal{O}$ is a cut where the oriented edges of the cut are directed consistently (but where some edges of the cut can be neutral). Let T_* be an ordered, q-rooted spanning tree of G. We say that $\mathcal{O} \in \mathcal{O}(G)$ is *cut positive* with respect to the data of (q, T_*) if in any potential cut $\mathrm{Cu} = \{U, V - U\}$ of $\mathcal{O}$ with $q \in V - U$ and $e_{\min} := \min^{<_{T_*}}(E(\mathrm{Cu}) \cap T_*)$ we have that $e^{\pm}_{\min} = (v, u) \notin \mathcal{O}$ where $u \in U$ and $v \in V - U$. In other words, a partial orientation is cut positive if the minimum edge (according to $<_{T_*}$) in any potential cut is either oriented from the part of the cut that does not contain q to the part that does or is neutral. Here when we say "minimum edge" we imagine that any edge not in T_* is greater than all edges in T_*. Denote the set of acyclic, cut positive partial orientations with respect to (q, T_*) by $\mathcal{A}^{(q,T_*)}$.

Conjecture 11 (See [5, Remark 4.40])**.** *Fix a pointed graph (G, q) and a total order $\prec^+$ on V^q. Then there exist an activity pair* $(\mathrm{in}, \mathrm{ex})$ *for G, an ordered, q-rooted spanning tree T_* of G, a total order $\prec^-$ on $V^q(G)$, and maps $\mathcal{O}_{\min}, \mathcal{O}_{\max}\colon F(G) \to \mathcal{O}(G)$ satisfying (1)-(8) of Theorem 4 as well as*

(9) $\mathcal{A}^{(q,T_*)}(G) = \bigsqcup_{F \in F(G), \mathrm{in}(F) \cap F = \emptyset} [\mathcal{O}_{\min}(F), \mathcal{O}_{\max}(F)]$;

(10) $\mathcal{A}^{(q,T_*)}(G) \cap \mathcal{A}^q(G) = \bigsqcup_{T \in T(G), \mathrm{in}(T) = \emptyset} [\mathcal{O}_{\min}(T), \mathcal{O}_{\max}(T)]$;

(11) $T \mapsto C^q(\mathcal{O}_{\max}(T))$ *bijects between* $\{T \in T(G)\colon \mathrm{in}(T) = \emptyset\}$ *and* $\mathrm{PF}^-(G, q, \prec^-)$.

Of course (10) in Conjecture 11 follows from (9) and (6), but we list it for emphasis. We now state the analogue of Corollary 5 for Conjecture 11. The proof of this corollary from Conjecture 11 is exactly the same as the proof of Corollary 5.

Corollary 12. *With the set-up of Conjecture 11, if that conjecture is true then:*

- $\sum_{\mathcal{O}\in\mathcal{A}^{(q,T_*)}(G)} x^{\#\Pi^{\prec^+}(\mathcal{O})} y^{\#\mathcal{O}} = x\, y^{n-1}\, (1+y)^g \cdot T_G(xy^{-1}(1+y),(1+y)^{-1})$;
- $\sum_{\mathcal{O}\in\mathcal{A}^q(G)\cap\mathcal{A}^{(q,T_*)}(G)} y^{\#\mathcal{O}} = y^{n-1}\, (1+y)^g \cdot T_G(0,(1+y)^{-1})$;
- $\mathrm{Hilb}(R/I^{-1,\prec^-}_{(G,q)}) = \sum_{c\in\mathrm{PF}^-(G,q,\prec^-)} y^{\deg(c)} = y^g \cdot T_G(0,1/y)$;
- $\{C^q(\mathcal{O})\colon \mathcal{O}\in\mathcal{A}^q(G)\cap\mathcal{A}^{(q,T_*)}(G)\} = \mathrm{PF}^-(G,q,\prec^-)$.

Example 13. Recall the set-up of Example 7. Take $T_* = \{e_1,e_2\}$ with $e_1 <_{T_*} e_2$ and $v_1 \prec^- v_2$. Note that $\{F\in F(G)\colon F\cap \mathrm{in}^<(F)=\emptyset\} = \{\emptyset,\{e_3\},\{e_2,e_3\}\}$. One can check in Figure 2 that the depicted interval decomposition indeed satisfies the additional requirements of Conjecture 11: for instance, the acyclic, cut positive partial orientations are precisely the ones in $[\mathcal{O}_{\min}(F),\mathcal{O}_{\max}(F)]$ for some forest $F\in\{\emptyset,\{e_3\},\{e_2,e_3\}\}$, and $\mathrm{PF}^-(G,q,\prec^-)=\{0\}$. △

Let us review the evidence in favor of Conjecture 11. The main evidence in favor of Conjecture 11 is that Backman-Hopkins [5] show

$$\sum_{\mathcal{O}\in\mathcal{A}^{(q,T_*)}(G)} y^{\#\mathcal{O}} = y^{n-1}\, (1+y)^g \cdot T_G(y^{-1}(1+y),(1+y)^{-1});$$
$$\sum_{\mathcal{O}\in\mathcal{A}^q(G)\cap\mathcal{A}^{(q,T_*)}(G)} y^{\#\mathcal{O}} = y^{n-1}\, (1+y)^g \cdot T_G(0,(1+y)^{-1});$$

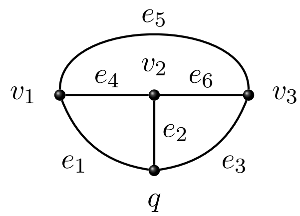
for any (G,q,T_*). (The elements of $\mathcal{A}^{(q,T_*)}(G)\cap\mathcal{A}^q(G)$ are called the *acyclic, cut positively connected* partial orientations with respect to the data of (q,T_*) in [5].) In other words, they show that the first bullet point of Corollary 12 (at least with $x:=1$) and the second bullet point always hold for all choices of data. Moreover, they conjecture [5, Conjecture 4.27] that for any pointed graph (G,q) there is always an ordered, q-rooted spanning tree T_* of G and $\prec^-$ a linear extension of $\preccurlyeq_{T_*}$ such that

- $\mathrm{Hilb}(R/I^{-1,\prec^-}_{(G,q)}) = y^g\cdot T_G(0,1/y)$;
- $\{C^q(\mathcal{O})\colon \mathcal{O}\in\mathcal{A}^q(G)\cap\mathcal{A}^{(q,T_*)}(G)\} = \mathrm{PF}^-(G,q,\prec^-)$.

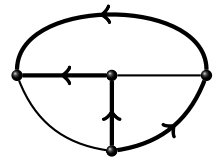
In other words, they conjecture that there is always some choice of data that satisfies the last two bullet points of Corollary 12. And in fact [5, Conjecture 4.27] is proved in some special cases in that paper: when G is saturated (appealing to Theorem 3), and when G is outerplanar.

But now we have to state some reasons to be skeptical about Conjecture 11. First of all, Desjardins gives an example [15, Example 21] that shows there is a pointed graph (G,q) and $\prec^-$ a total order on V^q such that $I^{-1,\prec^-}_{(G,q)}$ is well-defined but for which $\mathrm{Hilb}(R/I^{-1,\prec^-}_{(G,q)}) \neq y^g\cdot T_G(0,1/y)$. Moreover, things can go bad even when we restrict to the case of saturated G, as the following example shows. Note that this is a case where all of Corollary 12 holds (at least with $x:=1$ in the first bullet point; we take T_* to be a star rooted at q: see [5, Theorem 4.29]).

Example 14. Let $G = K_4$ be the complete graph on four vertices as below:



Take $T_* = \{e_1, e_2, e_3\}$ with $e_1 <_{T_*} e_2 <_{T_*} e_3$. Consider the partial orientation $\mathcal{O}$ below:



Then $\mathcal{O}$ is a minimal element of $\mathcal{A}^q(G) \cap \mathcal{A}^{(q,T_*)}(G)$. But $S(\mathcal{O})$ has a cycle, so $\mathcal{O}$ cannot be equal to $\mathcal{O}_{\min}(F)$ for any $F \in F(G)$. Thus no interval decomposition can satisfy the additional requirements of Conjecture 11 with this choice of T_*. $\triangle$

In light of Example 14 it would be very interesting to prove Conjecture 11 even in the special case where G is saturated. It seems that the main obstacle to proving (or disproving!) Conjecture 11 is that there are so many possible choices of data. Indeed, we may even have to move beyond the classical activity pair.

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