The coincidental down-degree expectations (CDE) phenomenon

LaCIM Combinatorics Seminar

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Section 1

Motivation from algebraic geometry

Brill-Noether theory: "representation theory for curves"

Let C be a (smooth, projective, algebraic) curve of genus g. Brill-Noether theory studies the collection $\mathcal{G}^r_d(C)$ of all maps $C \to \mathbb{P}^r$ of degree d. It turns out that $\mathcal{G}^r_d(C)$ is some kind of moduli "space" (e.g., algebraic variety or scheme). So we can study the geometry of $\mathcal{G}^r_d(C)$.

The *Brill-Noether number* is

$$\rho := \rho(g, r, d) := g - (r+1)(g - d + r).$$

Theorem (Griffiths-Harris, 1980)

When C is generic:

- *if ρ is nonnegative then $\mathcal{G}^r_d(C)$ has dimension ρ;*
- *if ρ is negative then $\mathcal{G}^r_d(C)$ is empty.*

From now on we will only consider generic C of genus g.

$\rho = 0$: Standard Young Tableaux

When $\rho = 0$, $\mathcal{G}^r_d(C)$ is a 0-dimensional variety, i.e., a finite set of points.

Theorem (Griffiths-Harris, 1980; Kempf, 1971; Kleiman-Laksov, 1972)

When $\rho = 0$, $\#\mathcal{G}^r_d(C) = g! \cdot \prod_{i=0}^{r} \frac{i!}{(g+d-r+i)!}$.

This is the same as the number of *Standard Young Tableaux* (SYTs) of rectangular shape $(r+1) \times (g-d+r)$, i.e., fillings of this rectangle with the numbers $1, \ldots, (r+1) \cdot (g-d+r)$, increasing in rows and columns.

Example

With $r = 1$, $g = 4$, $d = 3$ we have 2 maps and the SYTs are:

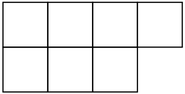
1	2
3	4

1	3
2	4

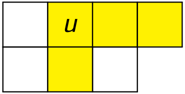
Hook-length formula for SYTs

We represent a partition λ as a set of boxes called a *Young diagram*:

$\lambda = (4, 3) \quad \Rightarrow$



For a box $u \in \lambda$, the *hook* of u is the set of boxes to the right or below u (including u); the *hook length* $h(u)$ is the number of boxes in this hook:



$h(u) = 4$

Theorem (Frame-Robinson-Thrall, 1954)

For λ a partition with number of boxes $|\lambda| = n$,

$$\#\mathrm{SYT}(\lambda) = n! \cdot \prod_{u \in \lambda} \frac{1}{h(u)}.$$

$\rho = 1$: Moduli space is a curve

Theorem (Fulton-Lazarsfeld, 1981)

If $\rho \geq 1$, $\mathcal{G}^r_d(C)$ is connected.

Theorem ("Gieseker-Petri theorem"; Gieseker, 1982)

$\mathcal{G}^r_d(C)$ is smooth.

Thus when $\rho = 1$, $\mathcal{G}^r_d(C)$ is a smooth curve. We can ask for finer geometrical information about $\mathcal{G}^r_d(C)$ in this case, such as its genus.

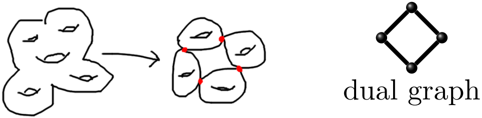
Theorem (Eisenbud-Harris, 1987; Pirola, 1985)

When $\rho = 1$, the genus of $\mathcal{G}^r_d(C)$ is $1 + \frac{(r+1)(g-d+r)}{g-d+2r+1} \cdot g! \cdot \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}$.

Recently Chan, López Martín, Pflueger, and Teixidor i Bigas (CLPT, 2018) developed a new combinatorial approach to this genus formula.

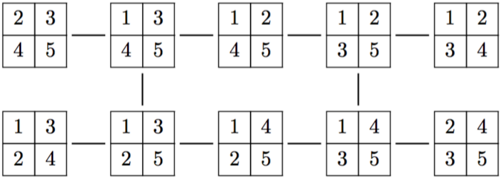
$\rho = 1$: "Almost-Standard Young Tableaux"

The first step of CLPT is to *degenerate* $\mathcal{G}^r_d(C)$ to a singular curve consisting of elliptic curves glued nodally:



The geometric information is now in the *dual graph* of the singular curve.

This dual graph is graph of *almost-SYTs* of shape $(r+1) \times (g-d+r)$, with two almost-SYTs connected by an edge when they differ in one box:



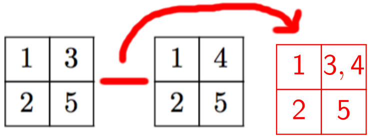
An *almost-SYT* of shape λ is like an SYT except that it can use $|\lambda| + 1$.

$\rho = 1$: "Standard Barely Set-Valued Tableaux"

The genus of $\mathcal{G}^r_d(C)$ is equal to the genus of the dual graph plus the sum of the genera of the components of the singular degeneration, which is:

$$(\#E - \#V + 1) + \#V = 1 + \#E$$

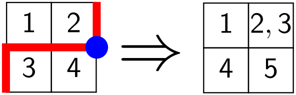
So CLPT want to count the number of edges of the dual graph. These edges correspond to *standard barely set-valued tableaux*, i.e., set-valued fillings of λ with the numbers $1, 2, \ldots, |\lambda| + 1$ each appearing exactly once, strictly increasing in rows and columns:



The genus formula is equiv. to claim that # of standard barely set-valued tableaux of shape $a \times b$ is $\frac{ab(ab+1)}{a+b}$ times # of SYT of shape $a \times b$.

$\rho = 1$: Expected number of corners

A standard barely set-valued tableaux of shape $a \times b$ is equivalent to the choice of (1) an SYT T of shape $a \times b$, (2) a lattice path from lower-left to upper-right of $a \times b$ "compatible" with T (entries to left of path are less than entries to right), and (3) a "leftward corner" of this lattice path:



Theorem (Main combinatorial theorem of CLPT, 2018)

Let $\mu_{\max}$ be the prob. distr. on lattice paths in $a \times b$ in which a path occurs proportional to number of SYTs with which it is compatible. Then the expected number of leftward corners for a path distr. $\sim \mu_{\max}$ is $\frac{ab}{a+b}$.

CLPT note that curiously this is the **same expected number** of corners as for the **uniform** distribution on lattice paths in $a \times b$.

$\rho > 1$: Higher dimensions

The natural generalization of "number of points" and "genus" to higher-dimensional varieties is *Euler characteristic*.

In follow-up work, Chan and Pflueger (preprint 2017) showed, for any ρ, the Euler characteristic of $\mathcal{G}^r_d(C)$ is (up to sign) the number of "standard set-valued tableaux" of shape $(r+1) \times (g-d+r)$ and content $\{1, \ldots, g\}$.

But for $\rho > 1$ there are no product formulas (although there are determinantal formulas- see Anderson-Chen-Tarasca, preprint 2017).

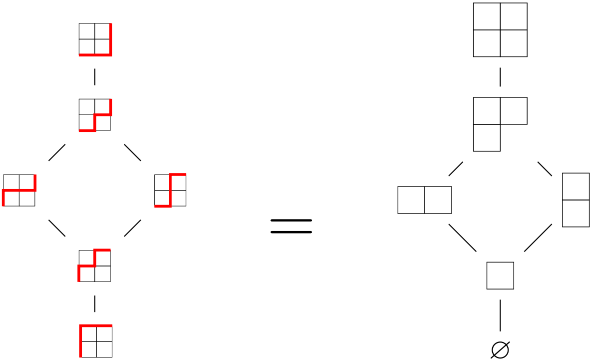
As a combinatorialist, what I find most interesting from all this Brill-Noether theory stuff are the (unexpected!) product formulas for certain tableaux. We will investigate how these formulas can be extended (to other shapes, etc). We will also see that CLPT's curious observation about the uniform distribution coincidence is more than just a curiosity...

Section 2

The coincidental down-degree expectations (CDE) property

Two probability distributions on a poset

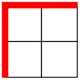
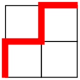
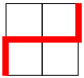
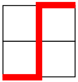
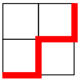
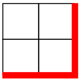
$L :=$ set of lattice paths in $a \times b$ rectangle, partially ordered by "stays weakly southeast of" ($\approx$ interval of Young's lattice of partitions):



Consider two distributions on L: the **uniform distribution** μ_{uni}; and the **maxchain distribution** $\mu_{\max}$ where $x \in L$ occurs proportional to the number of maximal chains ($\approx$ Standard Young Tableaux) containing x.

Expected down-degree

The *down-degree* of $x \in L$ is $\mathrm{ddeg}(x) := \#$ elem's x covers $(= \#$ corners$)$.

Lattice path x						
$\mu_{\max}(x)$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$
$\mu_{\mathrm{uni}}(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$\mathrm{ddeg}(x)$	0	1	1	1	2	1

Theorem (CLPT, 2018)

$\mathbb{E}(\mu_{\max}; \mathrm{ddeg})$ *is* $ab/(a+b)$.

Easy argument shows: $\mathbb{E}(\mu_{\mathrm{uni}}; \mathrm{ddeg})$ is also $ab/(a+b)$.

The CDE property

Now let L be any finite poset.

Definition (Reiner-Tenner-Yong (RTY), 2018)

We say L has the *coincidental down-degree expectations* property (or that "L is CDE") if $\mathbb{E}(\mu_{\mathrm{max}}; \mathrm{ddeg}) = \mathbb{E}(\mu_{\mathrm{uni}}; \mathrm{ddeg})$.

Note that $\mathbb{E}(\mu_{\mathrm{uni}}; \mathrm{ddeg})$ is the *edge-density* of L, i.e., number of edges of the Hasse diagram of L divided by number of elements of L.

RTY came up with this definition independently of the work of CLPT. They were studying tableaux in a more classical guise (Schubert calculus), which I will now explain...

Symmetric functions and cohomology of the Grassmannian

The *Grassmannian* $\mathrm{Gr}(k,n) :=$ space of k-planes in $\mathbb{C}^n$.
Natural stratification $\mathrm{Gr}(k,n) = \bigcup_{\lambda \subseteq k \times (n-k)} \sigma_\lambda$ into *Schubert cells*.

Its cohomology ring is well-understood in terms of this stratification:

$$\begin{aligned} H^*(\mathrm{Gr}(k,n)) &\simeq \{\text{symmetric polynomials in } x_1,\ldots,x_k\}/\langle h_{n-k+1},\ldots,h_n\rangle \\ [\overline{\sigma_\lambda}] &\mapsto s_\lambda(x_1,\ldots,x_k,0,0,\ldots), \end{aligned}$$

where s_λ are the *Schur functions* defined combinatorially by

$$s_\lambda = \sum_{\substack{\text{semistandard tableau } T \\ \text{of shape } \lambda}} \mathbf{x}^T.$$

Point for us: $[x_1 x_2 \cdots x_{|\lambda|}] s_\lambda = \#\ \mathrm{SYTs\ of\ shape}\ \lambda.$

Set-valued tableaux and K-theory of the Grassmannian

K-theory is a generalized cohomology theory. Study of the K-theory of the Grassmannian leads to analogs of Schur functions called *Grothendieck polynomials* and (under a suitable limit) *stable Grothendieck polynomials*.

Buch (2002) gave a combinatorial definition of the stable Grothendieck polynomials G_λ in terms of set-valued tableaux:

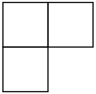
$$G_\lambda = \sum_{\substack{\text{semistandard set-valued} \\ \text{tableau } T \text{ of shape } \lambda}} (-1)^{|T|-|\lambda|} \mathbf{x}^T.$$

Note that s_λ is the lowest homogeneous component of G_λ.

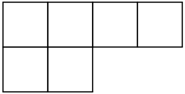
Point for us: $[x_1 x_2 \cdots x_{|\lambda|+1}] G_\lambda = (-1) \cdot \# \left(\begin{array}{c} \text{standard barely set-valued} \\ \text{tableaux of shape } \lambda \end{array} \right).$

Results and conjectures of Reiner-Tenner-Yong

RTY studied the CDE property for intervals of Young's lattice from the perspective of these symmetric functions.
Let $\delta_n := (n, n-1, \ldots, 1)$ denote the *staircase* partition:

$\delta_2 =$ 

and let $\delta_n \circ (a \times b)$ denote the *rectangular staircase* obtained from δ_n by replacing each box with $a \times b$ rectangle:

$\delta_2 \circ (1 \times 2) =$ 

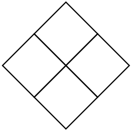
Theorem (Reiner-Tenner-Yong, 2018)

The interval $[\varnothing, \delta_n \circ (a \times b)]$ is CDE with edge-density $n(ab)/(a+b)$.

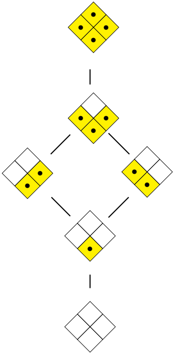
RTY also made a number of conjectures about CDE posets, most of which have been resolved using the *toggle perspective*, which I'll now explain...

Order ideals and distributive lattices

$P =$ finite poset. A subset $I \subseteq P$ is an *order ideal* if it's downwards-closed, i.e., if $y \in I, x \leq y \Rightarrow x \in I$. $J(P) :=$ poset of order ideals of P, ordered by containment (a.k.a. a *distributive lattice*).

$P =$ 

$[2] \times [2]$

$J(P) =$ 

We're interested in the CDE property for $L = J(P)$. Note for $I \in J(P)$, $\mathrm{ddeg}(I) = \#\max(I)$, and so the edge density of $J(P)$ is the average size of an *antichain* of P (subset of pairwise incomparable elements).

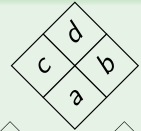
Toggling in a distributive lattice

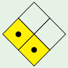
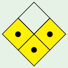
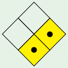
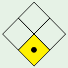
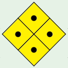
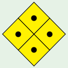
For $p \in P$, *toggling at p* is the involution $\tau_p \colon J(P) \to J(P)$:

$$\tau_p(I) := \begin{cases} I \cup \{p\} & \text{if } p \notin I \text{ and } I \cup \{p\} \in J(P); \\ I \setminus \{p\} & \text{if } p \in I \text{ and } I \setminus \{p\} \in J(P); \\ I & \text{otherwise.} \end{cases}$$

Toggling was popularized by Striker-Williams, 2012.

Example

$P =$ 

$\tau_b($  $) =$  $\tau_b($  $) =$  $\tau_b($  $) =$ 

Toggle-symmetric distributions and the tCDE property

We define toggleability statistics $\mathcal{T}_{p^+}, \mathcal{T}_{p^-} : J(P) \to \mathbb{Z}$:

$$\mathcal{T}_{p^\pm}(I) := \delta(p \text{ can be toggled into/out of } I).$$

Important: $\mathrm{ddeg}(I) = \sum_{p \in P} \mathcal{T}_{p^-}(I)$.

Definition

A probability distribution μ on $J(P)$ is *toggle-symmetric* if for all $p \in P$, $\mathbb{E}(\mu; \mathcal{T}_{p^+}) = \mathbb{E}(\mu; \mathcal{T}_{p^-})$. ("Can toggle in as often as out.")

Clear: μ_{uni} is toggle-symmetric (the toggles themselves show it).

Definition

$J(P)$ is *toggle CDE* (tCDE) if $\mathbb{E}(\mu; \mathrm{ddeg}) = \mathbb{E}(\mu_{\mathrm{uni}}; \mathrm{ddeg})$ for every toggle-symmetric distribution μ on $J(P)$.

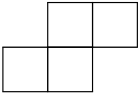
Lemma (Chan-Haddadan-Hopkins-Moci, 2017)

For any P, $\mu_{\max}$ on $J(P)$ is toggle-symmetric (so $J(P)$ tCDE $\Rightarrow$ CDE).

tCDE skew shapes

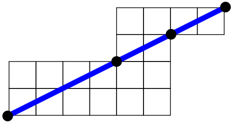
A *skew shape* λ/ν is set-theoretic difference of Young diagrams:

$$\lambda/\nu = (3,2)/(1,0) \quad \Rightarrow$$

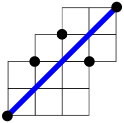


A *corner* of a skew shape is a point on the boundary where we "could add a box." Say λ/ν is *balanced* if all corners occur on main antidiagonal.

balanced:



not balanced:



Easy: rectangular staircase $\delta_n \circ (a \times b)$ is balanced.

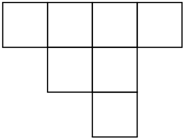
Theorem (Chan-Haddadan-Hopkins-Moci, 2017)

Let λ/ν be a balanced skew shape of height a, width b; then the interval $[\nu, \lambda]$ of Young's lattice is tCDE with edge density $ab/(a+b)$.

tCDE shifted shapes

A *strict partition* has strictly decreasing part sizes. We represent it by its *shifted Young diagram*:

$\lambda = (4, 2, 1)$ $\Rightarrow$

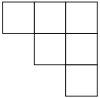


The *shifted Young's lattice* is poset of strict partitions ordered by containment of shifted Young diagrams. ("Type B/C" combinatorics.)

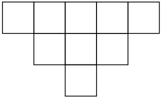
Theorem (Hopkins, 2017)

Let λ be a "shifted-balanced" shape; then the interval $[\varnothing, \lambda]_{\mathrm{shift}}$ of the shifted Young's lattice is tCDE with edge density $(\lambda_1 + 1)/4$.

"shifted-balanced" shape:



or



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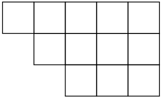
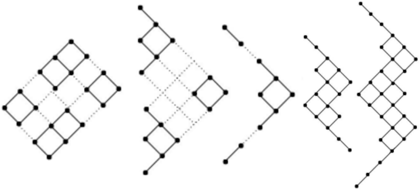
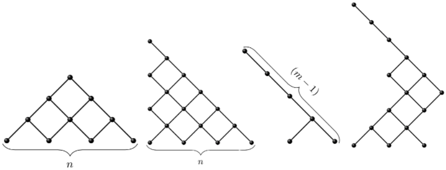


etc.

tCDE lattices from Lie theory



minuscule posets



root posets of coincidental type

Theorem (Case-by-case Hopkins, 2017; uniformly Rush preprint 2016)

Let P be a minuscule poset. Then $J(P)$ is tCDE.

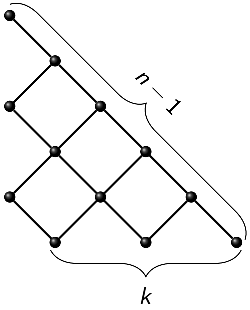
Theorem (Case-by-case Hopkins, 2019)

Let P be a root poset of coincidental type. Then $J(P)$ is tCDE.

(In all cases edge-density of $J(P)$ is $\#P/(\mathrm{rank}(P)+2)$.)

Difficult open case: trapezoid

The *trapezoid poset* $T_{k,n}$ is:



Conjecture (Reiner-Tenner-Yong, 2018)

$J(T_{k,n})$ is CDE.

N.B.: $J(T_{k,n})$ is **not** tCDE. So none of our techniques work!
Will be a project for participants of the UMN REU this summer...

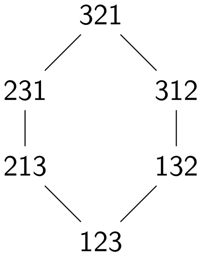
Weak order on the symmetric group

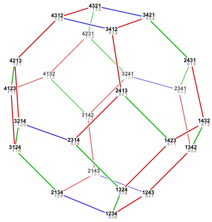
A *right inversion* of a permutation $w \in S_n$ is a pair (i, j) for $i < j$ with $w^{-1}(i) > w^{-1}(j)$ ("out of order values"); denote set of these by $\mathrm{Inv}^R(w)$:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix} \in S_5 \qquad \mathrm{Inv}^R(w) = \{(1,2),(1,5),(3,5),(4,5)\}$$

Weak order is the poset on S_n with $u \leq w$ iff $\mathrm{Inv}^R(u) \subseteq \mathrm{Inv}^R(w)$.

The Hasse diagram of weak order is 1-skeleton of the *permutohedron*:





Reduced words and cohomology of the flag variety

The *flag variety* is the space of *flags* of subspaces of $\mathbb{C}^n$:

$$\mathrm{Fl}_n := \{0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n = \mathbb{C}^n \colon \dim(V_i) = i\}.$$

Natural stratification $\mathrm{Fl}_n = \bigcup_{w \in S_n} \sigma_w$ into *Schubert cells*.

Study of cohomology of flag variety leads to analogs of Schur functions called *Schubert polynomials* (and *stable Schubert polynomials* F_w, etc.)

A *reduced word* of $w \in S_n$ is a minimal length factorization of w into adjacent transpositions:

$$w = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix} = (4,5) \circ (3,4) \circ (1,3) \circ (1,2)$$

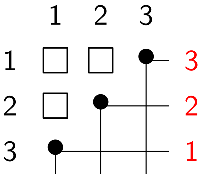
$$\text{reduced words of } w \Leftrightarrow \text{maximal chains of interval } [\mathrm{id}, w] \text{ of weak order}$$

Tableaux : Schur polynomials :: Reduced words : Schubert polynomials-

$$[x_1 x_2 \cdots x_{\ell(w)}] F_w = \# \text{ reduced words for } w$$

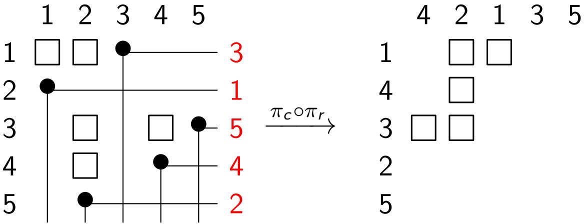
Dominant, vexillary, and skew vexillary permutations

A permutation $w \in S_n$ is *dominant of shape* λ if its *Rothe diagram* is λ:



$w = 321$
is dominant
of shape $(2, 1)$

A permutation $w \in S_n$ is *(skew) vexillary of (skew) shape* σ if its Rothe diagram can be transformed to σ by permuting rows and columns:



$w = 31542$
is skew vexillary
of shape
$(3, 2, 2)/(1, 1)$

tCDE intervals of weak order

Studying Schubert polynomials and their relatives, RTY proved:

Theorem (Reiner-Tenner-Yong, 2018)

Let w be a dominant permutation of shape $\delta_n \circ (a \times b)$; then the interval $[\mathrm{id}, w]$ of weak order is CDE with edge density $nab/(a+b)$.

Weak order is **not a distributive lattice**; but is a *semi*distributive lattice. Recent work of Reading, 2015; Barnard, 2019; and Thomas-Williams, preprint 2017, developed machinery to extend toggling from distributive to semidistributive lattices, allowing us to define a notion of tCDE for semidistributive lattices. This let us prove:

Theorem (Hopkins, preprint 2018)

Let w be a skew vexillary permutation of balanced shape with height a, width b; then $[\mathrm{id}, w]$ of is tCDE (and CDE) w/ edge density $ab/(a+b)$.

Section 3

Dynamical algebraic combinatorics

Rowmotion of order ideals

Now we relate CDE posets to "dynamical algebraic combinatorics."
Let $A(P)$ denote the set of antichains of P, and $F(P)$ the set of *order filters* (complements of order ideals). There are bijections:

$$J(P) \xrightarrow{\sim} F(P)$$
$$I \mapsto P \setminus I \qquad \text{(complement)}$$
$$F(P) \xrightarrow{\sim} A(P)$$
$$I \mapsto \min(I) \qquad \text{(minimal elements)}$$
$$A(P) \xrightarrow{\sim} J(P)$$
$$A \mapsto \{x \in P \colon x \leq y \text{ for some } y \in A\} \qquad \text{(down-set)}$$

The composition of these bijections is *rowmotion* $\mathrm{row}\colon J(P) \to J(P)$.

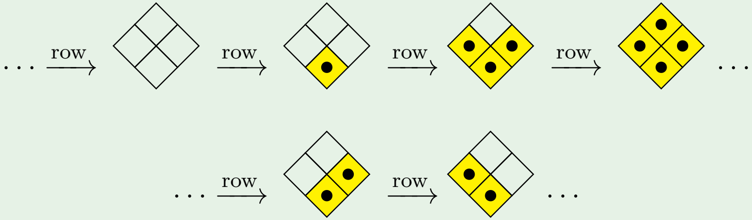
Theorem (Cameron–Fon-der-Flaass, 1995)

$\mathrm{row} = \tau_{p_1} \circ \tau_{p_2} \circ \cdots \circ \tau_{p_n}$, *where* $p_1, \ldots, p_n$ *is any linear extension of* P.
("Rowmotion is toggling top-to-bottom.")

Rowmotion example

Example

Consider rowmotion acting on $J([2] \times [2]) = [\varnothing, 2 \times 2]$. The two orbits of rowmotion are:



Observe that the order of rowmotion is 4, and that the average of $\mathrm{ddeg}(I) = \#\mathrm{max}(I)$ along each orbit is 1.

Theorem (Brouwer-Schrijver, 1974)

The order of row *on* $J(P)$ *for* $P = [a] \times [b]$ *is* $a + b$.

Antichain cardinality homomesy

Propp and Roby introduced the *homomesy phenomenon*.

Definition (Propp-Roby, 2015)

Let $\mathcal{S}$ be a combinatorial set, $\varphi\colon \mathcal{S} \to \mathcal{S}$ some invertible map, and $f\colon \mathcal{S} \to \mathbb{R}$ a combinatorial statistic. We say that f is c-mesic with respect to the action of φ on $\mathcal{S}$ if the average of f along every φ-orbit is c.

Homomesies are "dual" to invariant quantities. A motivating example was:

Theorem (Armstrong-Stump-Thomas, 2013; conj. by Panyushev)

The statistic $\#\mathrm{max}(I)$ is $n/2$-mesic w.r.t. row on $J(\Phi^+)$ for any root poset Φ^+, where $n =$ number of simple roots.

Concerning the example on the previous slide:

Theorem (Propp-Roby, 2015)

The statistic $\#\mathrm{max}(I)$ is $(ab)/(a+b)$-mesic w.r.t. row on $J([a]\times[b])$.

Homomesy and tCDE lattices

An observation of Striker connects rowmotion and tCDE posets:

Lemma (Striker, 2015)

For any $p \in P$, $\mathcal{T}_{p^+} - \mathcal{T}_{p^-}$ averages to zero along any row-orbit of $J(P)$.

Corollary

If $J(P)$ is tCDE, then $\#\max(I)$ is homomesic w.r.t. row on $J(P)$.

In this way we recover the Armstrong-Stump-Thomas antichain cardinality homomesy for the coincidental types, we recover the Propp-Roby antichain cardinality homomesy for the rectangle, and get much more...

Moreover, as we explain in the remaining slides, the connection between rowmotion homomesies and the tCDE property is “robust.”

Piecewise-linear and birational rowmotion

Einstein-Propp, 2014 defined *piecewise-linear* and *birational* rowmotion.

For any $f\colon P \to \mathbb{R}$, we define the *piecewise-linear toggle* $\tau_p^{\mathrm{PL}}(f)$ by

$$\tau_p^{\mathrm{PL}}(f)(r) := \begin{cases} f(r) & r \neq p; \\ \max_{q \lessdot p}\{f(q)\} + \min_{p \lessdot q}\{f(q)\} - f(p) & r = p, \end{cases}$$

with the convention $\max(\varnothing) = 0$ and $\min(\varnothing) = 1$. Combinatorial toggling is recovered by considering the indicator functions of order filters.

PL-rowmotion is given by a composition of all these PL-toggles from top-to-bottom:

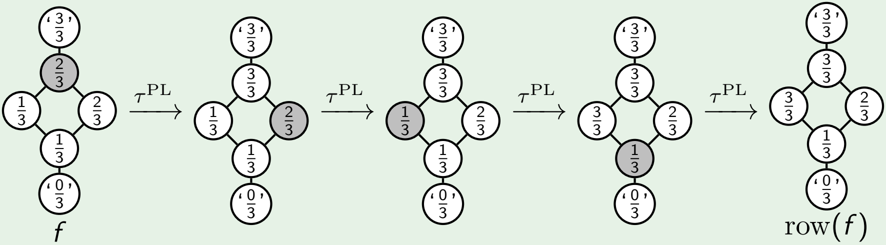
$$\mathrm{row}^{\mathrm{PL}} := \tau_{p_1}^{\mathrm{PL}} \circ \tau_{p_2}^{\mathrm{PL}} \circ \cdots \circ \tau_{p_n}^{\mathrm{PL}}$$

Birational rowmotion row^B is given by “detropicalizing” the PL expressions: turning $(\max, +)$ to $(+, \times)$.

Piecewise-linear rowmotion example

Example

Let $P = [2] \times [2]$. We compute:



Theorem (Grinberg-Roby, 2015)

The order of row^{PL} *(and* row^{B}*) for* $P = [a] \times [b]$ *is* $a + b$.

Proof based on Volkov's proof (2007) of Zamolodchikov periodicity.

Connections: cluster algebras; integrable systems; quiver representations...

Piecewise-linear and birational rowmotion homomesy

By defining PL-extensions of the toggleability statistics $\mathcal{T}_{p^\pm}$, we get a natural notion of antichain cardinality in the PL-realm (and then in the birational realm by detropicalizing):

$$\mathcal{T}_{p^-}^{\mathrm{PL}}(f) := \min_{p \lessdot q}\{f(q)\} - f(p);$$
$$\mathrm{ddeg}^{\mathrm{PL}}(f) := \sum_{p \in P} \mathcal{T}_{p^-}^{\mathrm{PL}}(f).$$

Can still ask about averages along orbits for PL-rowmotion (and "geometric averages" along orbits for birational rowmotion).

Theorem (Hopkins, preprint 2019)

Let P be such that $J(P)$ is tCDE; then the appropriate analogs of the antichain cardinality homomesy hold for PL- and birational rowmotion.*

*and P has a "2-dimensional grid-like structure" (all above examples do)

Thank you!

References:

- Chan, Haddadan, Hopkins, and Moci. "The expected jaggedness of order ideals." arXiv:1507.00249
- Reiner, Tenner, and Yong. "Poset edge densities, nearly reduced words, and barely set-valued tableaux." arXiv:1603.09589.
- Hopkins. "The CDE property for minuscule lattices." arXiv:1606.06248
- Rush. "On Order Ideals of Minuscule Posets III: The CDE Property." arXiv:1607.08018.
- Hopkins. "The CDE property for skew vexillary permutations." arXiv:1811.02404
- Hopkins. "Minuscule doppelgängers, the coincidental down-degree expectations property, and rowmotion." arXiv:1902.07301