Fourientations and the Tutte polynomial

Cornell Probability Seminar

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Section 1

Background: the Tutte polynomial and orientations

The Tutte Polynomial

The Tutte polynomial of a finite, undirected graph (but allowing multiple edges and loops) G is

$$T_G(x,y) = \sum_{U \subseteq E(G)} (x-1)^{\kappa(G_U)-\kappa(G)}(y-1)^{\kappa(G_U)+\#U-\#V(G)}$$

where $\kappa(\Gamma)$ is the number of connected components of Γ and G_U is the restriction of G to U.

If $G = G_1 \sqcup G_2$ then $T_G = T_{G_1} \cdot T_{G_2}$, so we will **assume all graphs are connected**: $\kappa(G) = 1$ for all G considered from now on.

The Tutte Polynomial in terms of activity

The previous definition makes it clear that $T_G(x, y)$ is a polynomial, but does not explain "what the Tutte polynomial is about." Tutte (1954) originally defined his polynomial as

$$T_G(x, y) = \sum_{i,j} t_{ij} x^i y^j$$

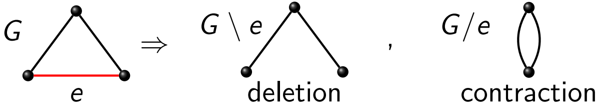
where t_{ij} is the number of spanning trees of G of "internal activity i and external activity j." Right now I won't explain what activity is, but the important part is **it depends on some extra decoration**:

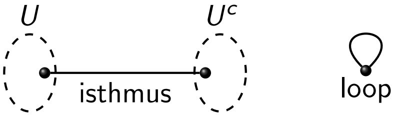
- Tutte (1954) used a total order $<$ on $E(G)$;
- Gessel-Sagan (1996) defined depth first search activity;
- Bernardi (2008) has a definition in terms of combinatorial maps;
- Julien Courtiel (2014) has a general framework for notions of activity.

Deletion-Contraction

From the activity definition of the Tutte polynomial it is not hard to deduce that $T_G = 1$ if G has no edges, and if $e \in E(G)$ then

$$T_G = \begin{cases} T_{G\setminus e} + T_{G/e} & \text{if } e \text{ is neither an isthmus nor a loop;} \\ xT_{G\setminus e} & \text{if } e \text{ is an isthmus;} \\ yT_{G/e} & \text{if } e \text{ is a loop.} \end{cases}$$





Weighted Deletion-Contraction and the Recipe Theorem

Any graph invariant satisfying a weighted deletion-contraction relation is essentially an evaluation of the Tutte polynomial:

Theorem (Recipe theorem, folklore?)

Suppose f is a $\mathbf{k}$-valued invariant of graphs such that $f(G) = 1$ if G has no edges, and for each G with at least one edge, there is some $e \in E(G)$ such that

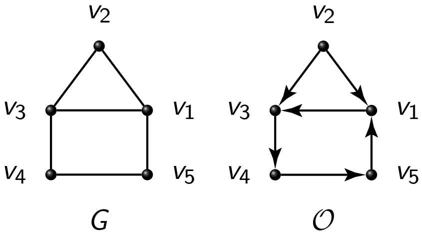
$$f(G) = \begin{cases} af(G/e) + bf(G \setminus e) & \textit{if } e \textit{ is neither an isthmus nor a loop;} \\ x_0 f(G \setminus e) & \textit{if } e \textit{ is an isthmus;} \\ y_0 f(G/e) & \textit{if } e \textit{ is a loop.} \end{cases}$$

Then $f(G) = a^{\#V(G)-1} b^{\#E(G)-\#V(G)+1} T_G(\frac{x_0}{a}, \frac{y_0}{b})$.

Set $n := \#V(G)$ and $g := \#E(G) - n + 1$.

Orientations

An *orientation* of a graph G is a choice for each edge $e = \{u, v\} \in E(G)$ of a direction (u, v) or (v, u). We treat $\mathcal{O}$ as a set of directed edges:

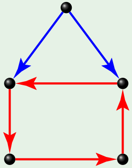


Here $\mathcal{O} = \{(v_1, v_2), (v_1, v_3), (v_1, v_5), (v_2, v_3), (v_3, v_4), (v_4, v_5)\}$. Since the seminal work of Stanley it has been known that the Tutte polynomial counts classes of graph orientations defined in terms of cuts and cycles.

Directed cycles (cuts) in orientations

A *directed cycle* of $\mathcal{O}$ is a cycle where the edges are oriented consistently. A *directed cut* of $\mathcal{O}$ is a partition of $V(G) = U \sqcup U^c$ such that all edges between U and U^c are directed from U^c to U.

Example



The red edges are a directed cycle. The blue edges are a directed cut.

An orientation is *acyclic* if it contains no directed cycles. An orientation is *strongly connected* if it contains no directed cuts.

The Tutte polynomial and orientations

- Stanley (1973) showed # acyclic orientations is $T_G(2,0)$.
- Las Vergnas (1980) showed # strongly connected $\mathcal{O}$'s is $T_G(0,2)$.
- Greene-Zaslavksy (1983) showed # of acyclic q-connected orientations is $T_G(1,0)$.
- Gioan (2007) showed # of indegree sequences among strongly connected orientations is $T_G(0,1)$.
- Stanley (1980) showed # of indeg. seq.'s among all $\mathcal{O}$'s is $T_G(2,1)$.
- Gioan (2007) showed # of q-connected $\mathcal{O}$'s is $T_G(1,2)$, and the number of their indegree sequences is $T_G(1,1)$.

Trivially $T_G(0,0) = 0$ (number of acyclic-strongly connected orientations) and $T_G(2,2) = 2^{|E(G)|}$ (total number of orientations).

The classical 3×3 square

Gioan (2007) unified all these results into a 3×3 square of orientation classes counted by $T_G(x, y)$ with $0 \leq x, y \leq 2$:

		Cut properties		
		General	q-connected	Strongly connected
Cycle properties	General	$T(2,2)$	$T(1,2)$	$T(0,2)$
	Inedg. seq.'s	$T(2,1)$	$T(1,1)$	$T(0,1)$
	Acyclic	$T(2,0)$	$T(1,0)$	$T(0,0)$

Bernardi (2008) connected the above 3×3 square to an analogous 3×3 square for classes of subgraphs (subsets of $E(G)$).

Section 2

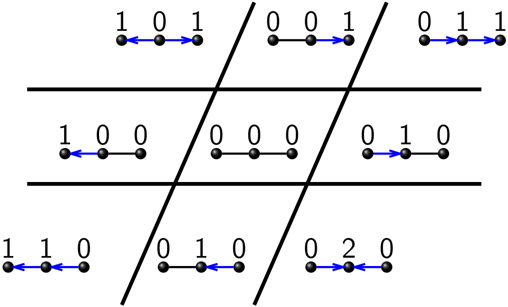
Bigraphical arrangements

Bigraphical arrangements

Let $A := (a_{e^\pm}) \in \mathbb{R}_{>0}^{2\#E(G)}$ be a *parameter list*. The *bigraphical arrangement* associated to A is the collection of $2\#E(G)$ hyperplanes:

$$\Sigma_G(A) := \{x_u - x_v = a_{e^\pm} \colon e^\pm = (u,v)\} \subseteq \mathbb{R}^n$$

There is a natural map $R \mapsto \mathcal{O}_R$ that sends a region of $\Sigma_G(A)$ to a *partial orientation* recording the third of each edge's "sandwich" the region is in:



The map $R \mapsto \mathrm{indeg}(\mathcal{O}_R)$ is the *Pak-Stanley* labeling of $\Sigma_G(A)$.

The G-Shi conjecture (theorem)

Let $G^\bullet$ denote the *cone over* G: $G^\bullet$ is obtained from G by adding a new vertex connected by an edge to each other vertex. Duval, Klivans and Martin (2011) conjectured that the Pak-Stanley labels of $\Sigma_G(A)$ for a certain choice of A are the $G^\bullet$*-parking functions*.

Note: The case $G = K_n$ recovers a famous bijection between the regions of the *Shi arrangement* and labeled trees due to Pak-Stanley (1996).

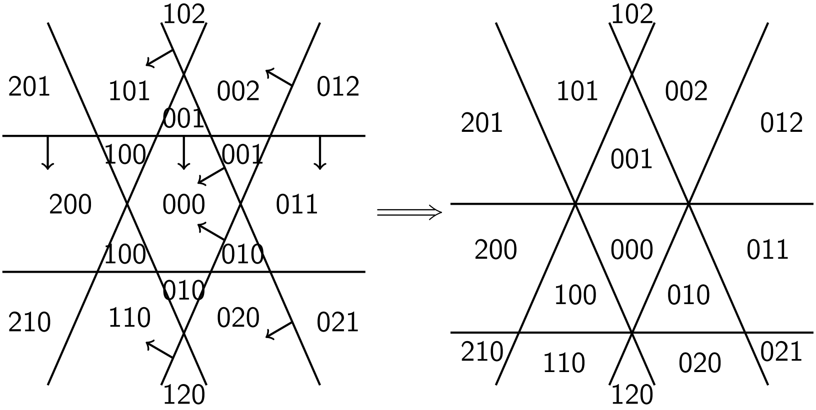
Theorem (H.-Perkinson, 2012)

For any $A \in \mathbb{R}_{>0}^{2\#E(G)}$, the set of Pak-Stanley labels of $\Sigma_G(A)$ is the set of $G^\bullet$-parking functions.

Note: As in the previous example, there may be duplicate labels.

"Sliding"

As hyperplanes slide, regions come and go but the set of Pak-Stanley labels remains the same as long as the central region is preserved:



G-parking functions

What are these G-parking functions? They are certain elements of $\mathbb{Z}V(G)$ satisfying some constraints (depending on a choice of *root* $q \in V(G)$) that I won't explain precisely. They are intimately related to *divisor theory (Riemman-Roch theory) for graphs*. A famous result of Merino gives the generating function for G-parking functions by degree in terms of the Tutte polynomial, where $\deg(c) := \sum_{v \in V(G)} c_v$ for $c = \sum_{v \in V(G)} c_v v \in \mathbb{Z}V(G)$.

Theorem (Merino, 1997)

We have

$$\sum_{c \text{ a } G\text{-parking function}} y^{\deg(c)} = y^g \cdot T_G(1, 1/y)$$

In particular their number is $T_G(1,1)$, the number of spanning trees of G.

Number of regions for generic parameters

In general it seems hard to compute the number of regions of $\Sigma_G(A)$, but in the case where the parameter list A is *generic* we get a generalized Tutte polynomial evaluation.

Theorem (H.-Perkinson, 2012)

Let $A \in \mathbb{R}_{>0}^{2\#E(G)}$ be generic. Then the number of regions of $\Sigma_G(A)$ is

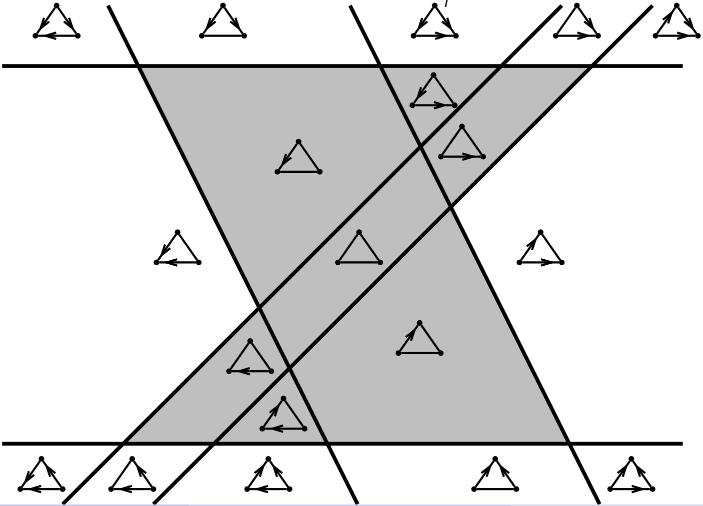
$$r(\Sigma_G(A)) = 2^{n-1} \cdot T_G(3/2, 1)$$

and the number of bounded regions is

$$b(\Sigma_G(A)) = 2^{n-1} \cdot T_G(1/2, 1).$$

Exponential parameters

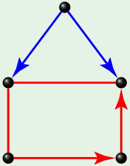
Let $<$ be a total order on the edges $E(G)$. If $e_1 < \cdots < e_m$, define the *exponential paramater list* $A^< := (a^<_{e^\pm})$ by $a^<_{e_i^\pm} := (1/2)^i$.



Potential/directed cycles (cuts) in partial orientations

A *potential cycle* (*cut*) of a partial orientation is the same as a directed cycle (cut) of a total orientation, except that some edges can be *neutral* (i.e., not oriented). On the other hand, a *directed cycle* (*cut*) of a partial orientation is as in a total orientation: only oriented edges are allowed.

Example



The red edges are a potential cycle. The blue edges are a directed cut.

Regions and cycle neutral partial orientations

A partial orientation such that the minimum edge (according to $<$) in each potential cycle is neutral is called *cycle neutral*. A partial orientation with no directed cuts is called *strongly connected*.

Proposition (Backman-H., 2015)

The map $R \mapsto \mathcal{O}_R$ is a bijection between the regions of $\Sigma_G(A^<)$ and the cycle neutral partial orientations of G. It restricts to a bijection between the bounded regions of $\Sigma_G(A^<)$ and the strongly connected-cycle neutral partial orientations of G.

(Thanks to Farbod Shokrieh for suggesting the use of exponential parameters at the AIM chip-firing workshop.)

Note: Exponential parameters are generic, so these classes of partial orientations are enumerated by the Tutte polynomial.

Section 3

Fourientations and min-edge classes

Fourientations

Last section: a class of partial orientations defined by **restrictions on the minimum edges of potential cycles** is enumerated by T_G.

Spencer and I substantially generalize this example. However, it turns out to be important to separate the neutral edges according to whether they can belong to potential cuts or potential cycles. A *fourientation* of G is therefore a choice for each $e \in E(G)$ whether to

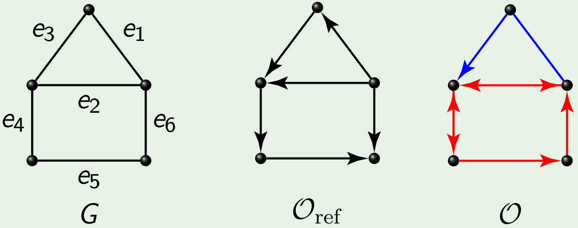
- orient e one way or another;
- leave e *unoriented* (these edges can belong to potential cuts);
- *biorient* e (these edges can belong to potential cycles).

There are $4^{\#E(G)}$ fourientations of a graph and thus the name.

We think of a fourientation $\mathcal{O}$ as a subset of $\mathbb{E}(G) := \{e^+, e^- \colon e \in E(G)\}$ where e^+ and e^- are edge orientations determined by some fixed reference orientation $\mathcal{O}_{\mathrm{ref}}$.

Fourientations example

Example



Here $\mathcal{O} = \{e_2^+, e_2^-, e_3^+, e_4^+, e_4^-, e_5^+, e_6^-\}$. The red edges are in a potential cycle and the blue edges are in a potential cut.

Min-edge classes

Definition

A *min-edge cycle class* is defined by a choice of $Y \subseteq \{\{+\},\{-\},\{+,-\}\}$. A potential cycle of $\mathcal{O}$ with minimum edge $e_{\min}$ is *bad* with respect to Y if

- $\{\delta\colon e^{\delta}_{\min} \in \mathcal{O}\} \in Y$;
- if $e_{\min}$ is bioriented in $\mathcal{O}$, then the cycle is directed against $e_{\min}$'s reference orientation.

The fourientation $\mathcal{O}$ is *good* w.r.t. Y if it has no bad potential cycles.

In other words, you can forbid the minimum edge $e_{\min}$ in a potential cycle from being oriented in agreement with $\mathcal{O}_{\mathrm{ref}}$, or oriented in disagreement with $\mathcal{O}_{\mathrm{ref}}$, or from being bioriented whenever that cycle is directed against $e_{\min}$'s reference orientation.

Min-edge cut classes are analogous.

The main theorem

Let $\mathcal{O}^o$ denote the oriented edges of $\mathcal{O}$, $\mathcal{O}^u$ the unoriented edges, and $\mathcal{O}^b$ the bioriented edges.

Theorem (Backman-H., 2015)

Let $X \subseteq \{\{+\},\{-\},\emptyset\}$, $Y \subseteq \{\{+\},\{-\},\{+,-\}\}$ be min-edge cut and cycle classes. Then we have

$$\sum_{\mathcal{O}} k^{\#\mathcal{O}^o} l^{\#\mathcal{O}^u} m^{\#\mathcal{O}^b} = (k+m)^{n-1}(k+l)^g T_G\left(\frac{x_0}{k+m},\frac{y_0}{k+l}\right)$$

where the sum is over good fourientations $\mathcal{O}$ of G w.r.t. X and Y, and

$$x_0 := \delta(\{+\}\notin X)k + \delta(\{-\}\notin X)k + \delta(\emptyset\notin X)l + m$$
$$y_0 := \delta(\{+\}\notin Y)k + \delta(\{-\}\notin Y)k + l + \delta(\{+,-\}\notin Y)m$$

where $\delta(P)$ is 1 if P is true and 0 if P is false.

Specializations

By specializing $(k, l, m) := (1, 1, 0)$ and $(k, l, m) := (1, 0, 1)$ in the main theorem we obtain enumerations for two families (Type A and Type B) of min-edge classes of partial orientations.

By specializing $(k, l, m) := (1, 0, 0)$ we obtain enumerations for min-edge classes of total orientations.

In this way we recover enumerations obtained by

- Gessel-Sagan (1996) for (acyclic) q-connected fourientations and partial orientations;
- Backman (2014) for acyclic, strongly connected, and cut/cycle minimal partial orientations;
- H.-Perkinson (2012) for the regions of $\Sigma_G(A)$ for generic A;
- all the authors for total orientation results mentioned earlier.

Fourientations

	General	Cut pos./neg. Cut directed	Cut neutral Cut (co)-con.	Cut internal
General	$2^{\|E\|}T(2,2)$	$2^{\|E\|}T(\frac{3}{2},2)$	$2^{\|E\|}T(1,2)$	$2^{\|E\|}T(\frac{1}{2},2)$
Cycle pos./neg. Cycle directed	$2^{\|E\|}T(2,\frac{3}{2})$	$2^{\|E\|}T(\frac{3}{2},\frac{3}{2})$	$2^{\|E\|}T(1,\frac{3}{2})$	$2^{\|E\|}T(\frac{1}{2},\frac{3}{2})$
Cycle neutral Cycle (co)-con.	$2^{\|E\|}T(2,1)$	$2^{\|E\|}T(\frac{3}{2},1)$	$2^{\|E\|}T(1,1)$	$2^{\|E\|}T(\frac{1}{2},1)$
Cycle external	$2^{\|E\|}T(2,\frac{1}{2})$	$2^{\|E\|}T(\frac{3}{2},\frac{1}{2})$	$2^{\|E\|}T(1,\frac{1}{2})$	$2^{\|E\|}T(\frac{1}{2},\frac{1}{2})$

Type A classes of partial orientations

	General	Cut pos./neg. Cut dir.	Cut neutral Cut (co)-con.	Cut int.
Gen.	2^g $T(3,\frac{3}{2})$	2^g $T(2,\frac{3}{2})$	2^g $T(1,\frac{3}{2})$	2^g $T(0,\frac{3}{2})$
Cycle min. Cycle max.	2^g $T(3,1)$	2^g $T(2,1)$	2^g $T(1,1)$	2^g $T(0,1)$
Acyc.	2^g $T(3,\frac{1}{2})$	2^g $T(2,\frac{1}{2})$	2^g $T(1,\frac{1}{2})$	2^g $T(0,\frac{1}{2})$

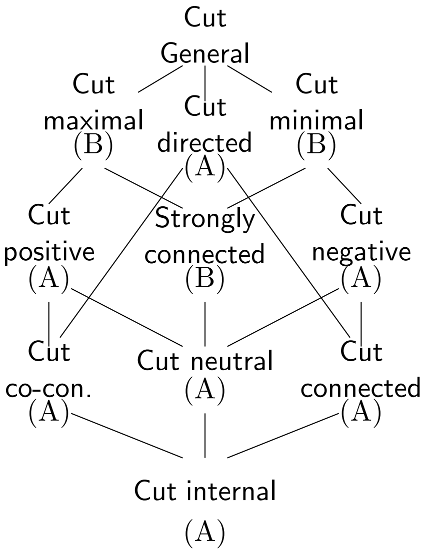
Type B classes of partial orientations

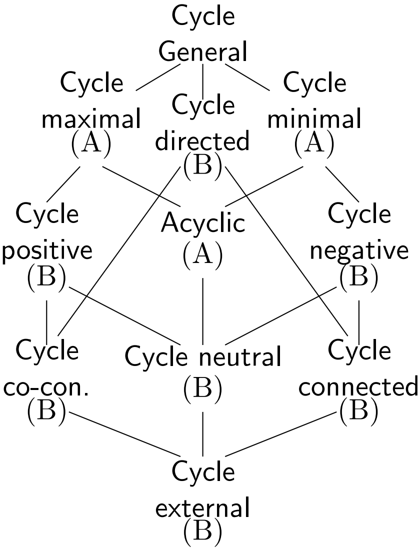
	General	Cut min./max.	Strong. con.
Gen.	2^{n-1} $T(\frac{3}{2},3)$	2^{n-1} $T(1,3)$	2^{n-1} $T(\frac{1}{2},3)$
Cycle pos./neg. Cycle dir.	2^{n-1} $T(\frac{3}{2},2)$	2^{n-1} $T(1,2)$	2^{n-1} $T(\frac{1}{2},2)$
Cycle neutral Cycle (co)-con.	2^{n-1} $T(\frac{3}{2},1)$	2^{n-1} $T(1,1)$	2^{n-1} $T(\frac{1}{2},1)$
Cycle ext.	2^{n-1} $T(\frac{3}{2},0)$	2^{n-1} $T(1,0)$	2^{n-1} $T(\frac{1}{2},0)$

Total orientations

	General	Cut min./max.	Strongly connected
General	$T(2,2)$	$T(1,2)$	$T(0,2)$
Cycle min./max.	$T(2,1)$	$T(1,1)$	$T(0,1)$
Acyclic	$T(2,0)$	$T(1,0)$	$T(0,0)$

Poset of min-edge partial orientation classes





Connections between min-edge classes and...

In our paper, Spencer and I explain how the min-edge classes of partial orientations and fourientations are related geometric, combinatorial, and algebraic topics such as:

- the bigraphical (and bicographical) arrangements;
- cycle/cocycle reversal systems (in the sense of Gioan and Backman);
- divisor theory (Riemann-Roch theory) for graphs;
- monomizations of power ideals (and zonotopal algebras);
- (co)graphic Lawrence ideals;
- the reliability polynomial.

As we saw in the case of the bigraphical arrangement, min-edge classes seem to arise in **situations where one cares about indegree sequences**.

Section 4

The reliability polynomial and cut connected fourientations

The reliability polynomial

Let $0 \leq p \leq 1$. Independently for each edge $e \in E(G)$, remove e from G with probability p. The *reliability polynomial* $R_G(p)$ is the probability that the resulting subgraph remains connected. It is well-known, and easy to see from the rank generating function expression for the Tutte polynomial (the first definition I gave), that

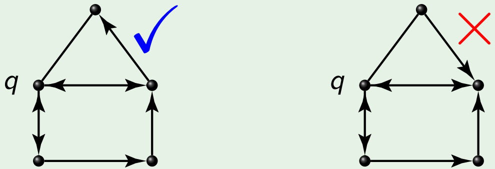
$$R_G(p) = (1-p)^{n-1}p^g \cdot T_G(1, 1/p).$$

Cut connected fourientations

A fourientation is *cut connected* if it belongs to the min-edge class defined by $X = \{\{-\}, \emptyset\}$. More explicitly, $\mathcal{O}$ is cut connected if every potential cut of $\mathcal{O}$ is directed in agreement with the reference orientation of the minimum edge of that cut.

Let $q \in V(G)$ be a choice of root. We can choose data $(\mathcal{O}_{\mathrm{ref}}, <)$ so that the cut connected fourienations are exactly the *q-connected fourientations*: the fourientations for which q is connected by a *potential path* (i.e., a directed path that can also use bioriented edges) to every $v \in V(G)$.

Example



Probability of being cut connected

Let k, l, m be nonnegative real numbers with $2k + l + m = 1$.
A (k, l, m)-fourientation is a randomly chosen fourientation where the probability of choosing $\mathcal{O}$ is $k^{\#\mathcal{O}^o} l^{\#\mathcal{O}^u} m^{\#\mathcal{O}^b}$.

Theorem (Backman-H., 2015)

The probability that a (k, l, m)-fourientation is cut connected is $R_G(p)$ where $p = k + l$.

The undirected and directed system reliability models

Let G_D be the *directed* graph obtained from G by including two directed edges (u, v),(v, u) for each $e = \{u, v\} \in E(G)$. Remove each directed edge from G_D independently with probability p; the previous theorem implies the probability the resulting "subdigraph" is q-connected is $R_G(p)$.

This result was recently obtained by Mohammadi (2015) in the context of combinatorial commutative algebra. (In fact, she shows that $p = p_e$ can depend on the edge e and we still have same probability of failure for the directed and undirected models.)

Problem

Give a short conceptual proof of why the undirected and directed system reliability models have the same probability of failure.

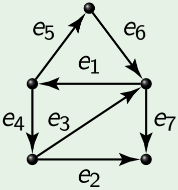
Section 5

Future work: fourientation activities

Orientation activities

Las Vergnas defines a notion of activity for total orientations. Continue to fix a total order $<$ on $E(G)$. An edge $e \in E(G)$ is *cut active* in $\mathcal{O}$ if it is the minimum edge in some directed cut of $\mathcal{O}$. An edge $e \in E(G)$ is *cycle active* in $\mathcal{O}$ if it is the minimum edge in some directed cycle. Let $I(\mathcal{O})$ denote the cut active edges of $\mathcal{O}$ and $L(\mathcal{O})$ the cycle active edges.

Example



Here $I(\mathcal{O}) = \{e_2\}$ and $L(\mathcal{O}) = \{e_1\}$.

The Las Vergnas formula

Let $\mathcal{O}_{\mathrm{ref}}$ be a reference orientation, and for a total orientation $\mathcal{O}$ let $\mathcal{O}^+$ be the edges of $\mathcal{O}$ oriented in agreement with $\mathcal{O}_{\mathrm{ref}}$ and $\mathcal{O}^-$ the edges oriented in disagreement with $\mathcal{O}_{\mathrm{ref}}$. Set $I(\mathcal{O}^+) := I(\mathcal{O}) \cap \mathcal{O}^+$ and so on.

Theorem (Las Vergnas, 1982/2012)

$$T_G(x+w, y+z) = \sum_{\mathcal{O}} x^{\#I(\mathcal{O}^+)} w^{\#I(\mathcal{O}^-)} y^{\#L(O^+)} z^{\#L(O^-)}$$

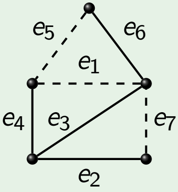
where the sum is over all total orientations $\mathcal{O}$ of G.

Note: Taking $x, w, y, z \in \{0, 1\}$ in this theorem of Las Vergnas recovers the classical 3×3 square of orientation enumerations.

Subgraph activities

There is an analogous story for subgraphs due to Gordon-Traldi. In this context, a *subgraph* of G is some subset of edges $S \subseteq E(G)$ together with all the vertices. An edge $e \in E(G)$ is *cut active* in S if it is the min. edge in some cut of **absent edges** in $S \setminus \{e\}$. An edge $e \in E(G)$ is *cycle active* in S if it is the min. edge in some cycle of **present edges** in $S \cup \{e\}$. Define $I(S)$ and $L(S)$ as for orientations.

Example



Here $I(S) = \{e_2\}$ and $L(S) = \{e_1\}$.

The Gordon-Traldi formula

Theorem (Gordon-Traldi, 1990)

$$T_G(x_* + w_*, y_* + z_*) = \sum_{S \subseteq E(G)} {x_*}^{\#I(S)\cap S} {w_*}^{\#I(S)\setminus S} {y_*}^{\#L(S)\setminus S} {z_*}^{\#L(S)\cap S}$$

Note: This theorem specializes to many other definitions of the Tutte polynomial:

- $x_* := 1$ and $y_* := 1$ gives the rank generating function expression;
- $w_* := 0$ and $z_* := 0$ gives Tutte's spanning tree activity expression.

Relationship between the two formulas

Obviously these are two formulas for the same Tutte polynomial evaluation (when the variables with stars equal those without). Giving a bijective proof of this fact that matches terms in the two sums is one aim of the so-called "active bijection" of Gioan-Las Vergnas (2009 and ongoing).

We offer a different connection between the two formulas: a fourientation activity formula that specializes to both. **Note** that a subgraph S is naturally a fourientation $\mathcal{O}_S$ where we treat the present edges as bioriented and the absent edges as unoriented.

In our perspective the active bijection is a self-map of fourientations.

Fourientation activities formula

Theorem (Backman-H.-Traldi, forthcoming)

There are fourientation cut activites $I(\mathcal{O})$ and cycle activities $L(\mathcal{O})$ with

$$(k+m)^{n-1}(k+l)^g T_G(\frac{kx+kw+mx_*+lw_*}{k+m}, \frac{ky+kz+ly_*+mz_*}{k+l}) =$$

$$\sum_{\mathcal{O}} k^{\#\mathcal{O}^o} l^{\#\mathcal{O}^u} m^{\#\mathcal{O}^b} x^{\#I(\mathcal{O}^+)} w^{\#I(\mathcal{O}^-)} {x_*}^{\#I(\mathcal{O}^b)} {w_*}^{\#I(\mathcal{O}^u)} y^{\#L(\mathcal{O}^+)} z^{\#L(\mathcal{O}^-)} {y_*}^{\#L(\mathcal{O}^u)} {z_*}^{\#L(\mathcal{O}^b)}$$

where the sum is over all fourientations $\mathcal{O}$ of G, and such that

- *setting $(k,l,m) := (1,0,0)$ recovers the Las Vergnas formula;*
- *setting $(k,l,m) := (0,1,1)$ recovers the Gordon-Traldi formula;*
- *setting $x_* := 1$, $y_* := 1$, and $x, w, w_*, y, z, z_* \in \{0,1\}$ recovers the main theorem for fourientation min-edge classes.*

Thank you!

these slides are available on my website