# Open Problems in Algebraic Combinatorics

blog submissions

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The rank and cranks

Submitted by Dennis Stanton

The Ramanujan congruences for the integer partition function $p(n)$ (see [1]) are

$$p(5n + 4) \equiv 0 \mod 5, \quad p(7n + 5) \equiv 0 \mod 7, \quad p(11n + 6) \equiv 0 \mod 11.$$ 

Dyson's rank [6] of an integer partition $\lambda = (\lambda_1, \lambda_2, \cdots)$

$$rank(\lambda) = \lambda_1 - \lambda'_1$$

(so that the rank is the largest part minus the number of parts) proves the Ramanujan congruences

$$p(5n + 4) \equiv 0 \mod 5, \quad p(7n + 5) \equiv 0 \mod 7$$

by considering the rank modulo 5 and 7.

**OPAC-001.** Find a 5-cycle which provides an explicit bijection for the rank classes modulo 5, and find a 7-cycle for the rank classes modulo 7.

The generating function for the rank polynomial is known to be

$$\sum_{n=0}^{\infty} rank_n(z)q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}.$$ 

The rank generating function $rank_{5n+4}(z)$ for partitions of $5n + 4$ does have an explicit factor of 5, but not positively. For example

$$rank_4(z) = 1 + z^{-3} + z^{-1} + z^3 + z^1 = (1 + z + z^2 + z^3 + z^4) \ast (1 - z + z^2)/z^3,$$

$$rank_{14}(z) = (1 + z + z^2 + z^3 + z^4) \ast p(z)/z^{13},$$

where $p(z)$ is an irreducible polynomial of degree 22 which has negative coefficients. For an explicit 5-cycle which would be a rank class bijection, one would expect the factor
1 + z + z^2 + z^3 + z^4 times a positive Laurent polynomial in z. Here is a conjectured modification that does this.

**Definition.** For \( n \geq 2 \) let

\[
M_{\text{rank}}(z) = \text{rank}_n(z) + (z^{n-2} - z^{n-1} + z^{2-n} - z^{1-n}).
\]

**OPAC-002.** For \( n \geq 0 \) show that the following are non-negative Laurent polynomials in \( z \):

\[
M_{\text{rank}}_{5n+4}(z)/(1 + z + z^2 + z^3 + z^4),
M_{\text{rank}}_{7n+5}(z)/(1 + z + z^2 + z^3 + z^4 + z^5 + z^6).
\]

This conjecture says that the rank definition only needs to be changed for \( \lambda = n, 1^n \to \) have the “correct” symmetry. I do not know a modification which will also work modulo 11. Frank Garvan has verified OPAC-002 for \( 5n + 4 \leq 1000 \) and \( 7n + 5 \leq 1000 \).

The Andrew-Garvan [2] *crank* of a partition \( \lambda \) is

\[
AG_{\text{crank}}(\lambda) = \begin{cases}
\lambda_1 & \text{if } \lambda \text{ has no } 1's \\
\mu(\lambda) - (#1's \text{ in } \lambda) & \text{if } \lambda \text{ has at least one } 1,
\end{cases}
\]

where \( \mu(\lambda) \) is the number of parts of \( \lambda \) which are greater than the number of \( 1 \)'s of \( \lambda \). For example

\[
AG_{\text{crank}}(1111) = 0 - 4, \quad AG_{\text{crank}}(211) = 0 - 2, \quad AG_{\text{crank}}(22) = 2 - 0
\]
\[
AG_{\text{crank}}(31) = 1 - 1, \quad AG_{\text{crank}}(4) = 4 - 0.
\]

The generating function of the AGcrank over all partitions of \( n \) is \( AG_{\text{crank}}_n(z) \). For example

\[
AG_{\text{crank}}_4(z) = z^{-4} + z^{-2} + z^2 + z^0 + z^4.
\]

The generating function for the AGcrank polynomial is known to be (after modifying \( AG_{\text{crank}}_1(z) \))

\[
\sum_{n=0}^{\infty} AG_{\text{crank}}(z)q^n = \frac{(q; q)_\infty}{(zq; q)_\infty (q/z; q)_\infty}.
\]

**OPAC-003.** Show

\[
AG_{\text{crank}}_{5n+4}(z) = (1 + z^2 + z^4 + z^6 + z^8)*\text{(a positive Laurent polynomial in } z)\).
\]
Frank Garvan has verified OPAC-003 for $5n + 4 \leq 1000$.

Ramanujan factored the first 21 AGcrank polynomials, $\lambda_n = AGcrank_n(a)$, see the paper of Berndt, Chan, Chan and Liaw [5, p. 12]. Ramanujan found the factor

$$\rho_5 = z^4 + z^{-4} + z^2 + z^{-2} + 1$$

for $n = 4, 9, 14, 19$ but the other factors did not always have positive coefficients. For example Ramanujan had

$$AGcrank_{14}(z) = (z^4 + z^2 + 1 + z^{-2} + z^{-4}) \ast \rho_9 \ast (a_5 - a_3 + a_1 + 1),$$

where

$$\rho_9 \ast (a_5 - a_3 + a_1 + 1) = (z^2 + z^{-2} + 1)(z^3 + z^{-3} + 1) \ast (z^3 + z^{-5} - z^3 - z^{-3} + z + z^{-1} + 1)$$

$$= 3 + 1/z^{10} + 1/z^7 + 1/z^6 + 1/z^5 + 2/z^4 + 2/z^3 + 2/z^2 + 2/z$$

$$+ 2z + 2z^2 + 2z^3 + 2z^4 + z^5 + z^6 + z^7 + z^{10}.$$

A modified version of the AGcrank works for modulo 5, 7, and 11, with only the values at partitions $n, 1^n$ changed.

**Definition.** For $n \geq 2$ let

$$MAGcrank_{n,a}(z) = AGcrank_n(z) + (z^{n-a} - z^n + z^{a-n} - z^{-n}).$$

**OPAC-004.** Show that the following are non-negative Laurent polynomials in $z$

$$MAGcrank_{5n+4,5}(z)/(1 + z + z^2 + z^3 + z^4),$$

$$MAGcrank_{5n+5,7}(z)/(1 + z + z^2 + z^3 + z^4 + z^5 + z^6),$$

$$MAGcrank_{11n+6,11}(z)/(1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9 + z^{10}).$$

Frank Garvan has verified OPAC-004 for $t_n + r \leq 1000$.

The 5-corecrank (see [7]) may be defined from the integer parameters $(a_0, a_1, a_2, a_3, a_4)$ involved in the 5-core of a partition $\lambda$. Its generating function for partitions of $5n + 4$ is

$$\sum_{n=0}^{\infty} q^{n+1} \sum_{\lambda \vdash 5n+4} z^{5\text{corecrank}(\lambda)} = \frac{1}{(q; q)_{\infty}^5} \sum_{\bar{a}, \sum_{i=0}^{4} i a_i} q^{Q(\bar{a})} z^{\sum_{i=0}^{4} i a_i}$$

where

$$Q(\bar{a}) = \sum_{i=0}^{4} a_i^2 - \sum_{i=0}^{4} a_i a_{i+1}, \quad a_5 = a_0.$$
Bijections for the core crank classes are known [7] for 5, 7, and 11.

Frank Garvan noted the following version of the previous conjectures holds for the 5corecrank for \( n \leq 100 \), and \( n \leq 8 \) see [3].

**OPAC-005.** Show that the following are non-negative Laurent polynomials in \( z \)

\[
5\text{corecrank}_{5n+4}(z)/(1 + z + z^2 + z^3 + z^4),
\]
\[
5\text{corecrank}_{5n+4,j}(z)/(1 + z + z^2 + z^3 + z^4) \text{ when restricted to } BG\text{crank} = j.
\]

Bringmann, Ono, and Rolen [8] have proven the first statement.

References:


Sorting via chip-firing

Submitted by James Propp

This post concerns various dynamical systems whose states are configurations of labeled chips on the one-dimensional integer lattice. In these configurations, multiple chips can occupy the same site on the lattice (and the “relative” position of chips at the same site is irrelevant).

The main result of [3] is the following theorem:

**Theorem.** Put chips labeled \(1\) through \(n\) at site 0 on the integer lattice, and repeatedly apply moves of the form

\[
\text{If chips } (a) \text{ and } (b) \text{ are both at site } k \text{ with } a < b \text{ then slide chip } (a) \text{ to site } k - 1 \text{ and chip } (b) \text{ to site } k + 1
\]

until no further moves can be performed. Then if \(n \equiv 0 \mod 2\), the final configuration of the chips is independent of the moves that were made, and in particular, the chips are sorted in the sense that if \(a < b\) then chip \((a)\) is to the left of chip \((b)\).

Note that if \(n \equiv 1 \mod 2\) there can be multiple final configurations and the chips need not end up sorted.

The following conjectures are variants of the above theorem.

**OPAC-006.** Put chips labeled \(1\) through \(n\) at site 0 on the integer lattice, and repeatedly apply moves of the form

\[
\text{If chips } (a), (b) \text{ and } (c) \text{ are all at site } k \text{ with } a < b < c, \text{ then slide chip } (a) \text{ to site } k - 1 \text{ and chip } (c) \text{ to site } k + 1
\]

until no further moves can be performed. Show that if \(n \equiv 3 \mod 4\), then the final configuration of the chips is independent of the moves that were made, and in particular, the chips are weakly sorted in the sense that if \(a < b\) then chip \((a)\) is not to the right of chip \((b)\).

Note: This is a special case of Conjecture 22 from [3].
OPAC-007. Put chips labeled \((1)\) through \((n)\) at site 0 on the integer lattice, and repeatedly apply moves of the form

If chips \((a)\) \((b)\) \((c)\) and \((d)\) are all at site \(k\) with \(a < b < c < d\) then slide chips \((a)\) and \((b)\) to site \(k - 1\) and chips \((c)\) and \((d)\) to site \(k + 1\)

until no further moves can be performed. Show that if \(n \equiv 0 \mod 4\) then the final configuration of the chips is independent of the moves that were made, and in particular, the chips are weakly sorted.

Note: This is a special case of Conjecture 25 from [3].

It is possible that the methods of that paper could with effort be made to solve these problems. However, I would much rather see a new and simpler approach (perhaps using the connection to root systems explored in [1] and [2]).

References:


On the cohomology of the Grassmannian

Submitted by Victor Reiner

The $Q$-binomial coefficient is defined as

$$\binom{k + \ell}{k}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q}$$

where $[n]!_q = [n]_q [n-1]_q \cdots [2]_q [1]_q$ with $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$. It has many interpretations: combinatorial, algebraic, and geometric. For example, it is the Hilbert series for a graded ring that we will call here $R^{k,\ell}$, the cohomology ring with rational coefficients for the Grassmannian $\text{Gr}(k, \mathbb{C}^{k+\ell})$ of $k$-planes in $\mathbb{C}^{k+\ell}$, with grading rescaled by half:

$$\binom{k + \ell}{k}_q = \text{Hilb}(R^{k,\ell}, q) := \sum_{d \geq 0} q^d \dim_{\mathbb{Q}}(R^{k,\ell})_d$$

$$= \sum_{d=0}^{k\ell} q^d \dim_{\mathbb{Q}} H^{2d}(\text{Gr}(k, \mathbb{C}^{k+\ell}), \mathbb{Q}).$$

We know plenty about the structure of this ring. For example, it can be presented as the quotient of the ring of symmetric functions in infinitely many variables by the $\mathbb{Q}$-span of all Schur functions $s_\lambda$ for which $\lambda$ does not lie in a $k \times \ell$ rectangle $(\ell^k)$. Thus it has a $\mathbb{Q}$-basis given by $\{s_\lambda\}_{\lambda \subseteq (\ell^k)}$ and its multiplicative structure constants in $s_\mu s_\nu = \sum_\lambda c_{\mu\nu}^\lambda s_\lambda$ are the well-understood Littlewood-Richardson coefficients, interpreting $c_{\mu\nu}^\lambda = 0$ if $\lambda \not\subseteq (\ell^k)$. On the other hand, it also has at least two simple presentations via generators and relations:

$$R^{k,\ell} \cong \mathbb{Q}[e_1, e_2, \ldots, e_k, h_1, h_2, \ldots, h_\ell] / \left( \sum_{i+j=d} (-1)^i e_i h_j \right)_{d=1,2,\ldots,k+\ell}$$

$$\cong \mathbb{Q}[e_1, e_2, \ldots, e_k] / (h_{\ell+1}, h_{\ell+2}, \ldots, h_{\ell+k})$$
where in the second line, $h_r$ can be computed via (dual) Jacobi-Trudi determinants:

$$
h_1 = |e_1| = e_1, \quad h_2 = \begin{vmatrix} e_1 & e_2 \\ e_1 & 1 \end{vmatrix} = e_1^2 - e_2, \quad h_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ 1 & e_1 & e_2 \\ 0 & 1 & e_1 \end{vmatrix} = e_1^3 - 2e_1e_2 + e_3, \text{ etc.}
$$

Since $R^{k,\ell} \cong R^{\ell,k}$, we will assume from now on that $k \leq \ell$. The open problem here is to understand the Hilbert series for a tower of graded subalgebras

$$
\mathbb{Q} = R^{k,\ell,0} \subset R^{k,\ell,1} \subset \cdots \subset R^{k,\ell,k} = R^{k,\ell},
$$

where $R^{k,\ell,m}$ is the $\mathbb{Q}$-subalgebra of $R^{k,\ell}$ generated by all elements of degree at most $m$: that is, the subalgebra generated by $e_1, \ldots, e_m$. Note for $m = 0$, it is silly, as $R^{k,\ell,0} = \mathbb{Q}$ so $\text{Hilb}(R^{k,\ell,0}, q) = 1$.

The $m = 1$ case is less silly. Here it turns out that

$$
R^{k,\ell,1} \cong \mathbb{Q}[e_1]/(e_1^{k\ell+1}).
$$

It is no surprise that $R^{k,\ell,1}$ would be a truncated polynomial algebra in the generator $e_1$. It was less clear why the last nonvanishing power would be $e_1^{k\ell}$, matching the top nonvanishing degree in $R^{k,\ell}$. This follows either from

- a direct calculation with the Pieri formula showing $e_1^{k\ell} = f^{(k\ell)} s_{(k\ell)}$ where $f^{\lambda}$ is the (nonzero!) number of standard Young tableaux of shape $\lambda$, or
- by a special case of the Hard Lefschetz Theorem, since $e_1$ represents the cohomology class dual to a hyperplane section of the smooth variety $G_T(k, \mathbb{C}^{k+\ell})$ in its Plücker embedding.

As a consequence,

$$
\text{Hilb}(R^{k,\ell,1}, q) = 1 + q + q^2 + \cdots + q^{k\ell},
$$

or equivalently, the filtration quotient $R^{k,\ell,1}/R^{k,\ell,0}$ has Hilbert series

$$
\text{Hilb}(R^{k,\ell,1}/R^{k,\ell,0}, q) = \text{Hilb}(R^{k,\ell,1}, q) - \text{Hilb}(R^{k,\ell,0}, q) = q + q^2 + \cdots + q^{k\ell} = q \cdot [k\ell]_q = q \cdot [\ell]_q \cdot [k]_q^{\ell}.
$$
For small $k$, $\ell$, $m$ one can compute $\text{Hilb}(R^{k,\ell,m}, q)$ via computer algebra, e.g., Macaulay2. One can first find a presentation for $R^{k,\ell,m}$ in terms of the generators $e_1, \ldots, e_m$ using Gröbner basis calculations that eliminate the variables $e_{m+1}, e_{m+2}, \ldots, e_k$ from the above presentations. Given such a presentation, computer algebra lets one then compute the Hilbert series.

After doing this for small $k$, $\ell$, $m$, one quickly notices that the Hilbert series for the filtration quotients $R^{k,\ell,m}/R^{k,\ell,m-1}$ are not only divisible by $q^m$, as forced by their definition, but also divisible by $\left[ \ell \atop m \right] q$, which is not a priori obvious. Dividing these factors out leads to this conjecture.

**OPAC-008.** For integers $m$ and $\ell$ with $1 \leq m \leq \ell$, does the following hold?

\[
\text{Hilb}(R^{k,\ell,m}/R^{k,\ell,m-1}, q) = q^m \cdot \left[ \ell \atop m \right] q \sum_{j=0}^{k-m} q^{j(m+1)} \left[ m+j-1 \atop j \right]_q.
\]

For example, when $m = 1$, OPAC-008 predicts what we saw above:

\[
\text{Hilb}(R^{k,\ell,1}/R^{k,\ell,0}, q) = q^1 \cdot \left[ \ell \atop 1 \right] q \sum_{j=0}^{k-1} q^{j\ell} \left[ j \atop j \right] q = q \cdot [\ell]_q \cdot [k]_q.
\]

Geanina Tudose and I were led to OPAC-008 after realizing that one of its much weaker implications [2, Conjecture 4] about $\text{Hilb}(R^{k,\ell,m}, q)$ would greatly simplify the proof of the following interesting result of Hoffman, on graded endomorphisms of the cohomology ring $R^{k,\ell}$.

**Theorem (Hoffmann [1]).** Let $\varphi$ be a graded algebra endomorphism $\varphi$ of $R^{k,\ell}$ that scales $R^{1,\ell}_1$ via some nonzero $\alpha$ in $\mathbb{Q}$. If $k \neq \ell$, then $\varphi$ scales each component $R^{k,\ell}_d$ via $\alpha^d$. If $k = \ell$, then $\varphi$ has the form just described, or its composition with the involution swapping $e_r \leftrightarrow h_r$ for all $r$.

This theorem was conjectured by O’Neill without the assumption that $\alpha$ is nonzero, motivated by a topological application: assuming it, one can easily apply the Lefschetz fixed point theorem to show $G_1(k, \mathbb{C}^{k+\ell})$ has the fixed point property (i.e. every continuous self-map has a fixed point) if and only if $k \neq \ell$ and $k, \ell$ is even.

In [2], one finds more background on the OPAC-008, including verification of the case $m = k$, and how the conjecture would shorten Hoffman’s proof from ten pages to two pages. Here are a few more remarks.

**Remark 1.** Naming the inner sum in OPAC-008 as
for $0 \leq m \leq k \leq \ell$, one can check that it is defined by this recurrence

$$f_{m}^{k, \ell}(q):=\sum_{j=0}^{k-m} q^{j(\ell-m+1)} \left[\begin{array}{l} m+j-1 \end{array}\right]_{q}$$

and initial conditions $f_{0}^{k, \ell}(q) = f_{k}^{k, \ell}(q) = 1$. Thus $f_{m}^{k, \ell}(q)$ is a $q$-analogue of the binomial coefficient $\binom{k}{m}$ which depends on $\ell$, and has a different $q$-Pascal recurrence than that of $\binom{k}{m}$.

**Remark 2.** One might approach OPAC-008 by finding $\mathbb{Q}$-bases of $R^{k, \ell}$ that respect the filtration by $R^{k, \ell, m}$, through Gröbner basis calculations with a lexicographic term order with $e_{k} > \cdots > e_{2} > e_{1}$ and understanding the structure of the standard monomials. We have so far failed to make this work!

**Remark 3.** Recall that $R^{k, \ell}$ is the quotient of the ring of symmetric functions $\Lambda$ by a certain ideal, and $R^{k, \ell, m}$ is the subalgebra of $R^{k, \ell}$ generated by its degrees up to $m$. The $m$-Schur functions give a $\mathbb{Q}$-basis for the subalgebra $\Lambda_{(m)}$ of $\Lambda$ generated by its degrees up to $m$. Perhaps there is a convenient subset of $m$-Schur functions, for varying values of $m$, whose images in $R^{k, \ell}$ give a $\mathbb{Q}$-basis respecting the filtration by $R^{k, \ell, m}$?

**Remark 4.** It is well-known how to generalize the cohomology ring $R^{k, \ell} = H^{*}(\text{Gr}(k, \mathbb{C}^{k+\ell}, \mathbb{Q})$ in many directions: to other flavors of cohomology (quantum, equivariant, etc.), to other partial flag manifolds in type $A$, and to other Lie types. Perhaps one should approach OPAC-008 by first generalizing it in one of these directions?

References:


The Schur cone and the cone of log concavity

Submitted by Dennis White

Let $C^k_N$ be the cone generated by products (homogeneous of degree $N$) of Schur functions $s_{\lambda}$, where $l(\lambda) \leq k$. That is, $C^k_N$ is the set of all non-negative linear combinations of vectors of the form

$$s_{\lambda_1}s_{\lambda_2} \cdots s_{\lambda^n}$$

where $l(\lambda^i) \leq k$ and where $|\lambda^1| + \cdots + |\lambda^n| = N$

We call this cone the $k$-Schur cone of degree $N$. Our goal is to find the extreme vectors of this cone. That is, we wish to find products of Schur functions as above which cannot be written as a positive linear combinations of other products of that form. For instance, since $s_{3,1}s_2 = s_{3,2}s_1 + s_{1,2}s_4$, we know $s_{3,1}s_2$ is not extreme in $C^2_6$.

We can dispense with two easy cases immediately. When $k = 1$, since $h_i = s_{\nu}C^1_N$ is then the cone generated by $h_{\lambda} \lambda \vdash N$. Since the $h_{\lambda}$ are a basis, none can be written as a linear combination of the others, so the $h_{\lambda} \lambda \vdash N$, will be extreme.

When $k = N$, $C^N_N$ is the cone generated by products of Schur functions. But by the Littlewood-Richardson rule, products of Schur functions are positive linear combinations of Schur functions, so $s_{\lambda} \lambda \vdash N$ will be extreme.

When $k = 2$, the Jacobi-Trudi identity says the cone is generated by products of the form

$$h_i h_j - h_{i+1} h_{j-1} \quad \text{and} \quad h_i \quad i \geq j \geq 1.$$

We therefore call the 2-Schur cone of degree $N$ the cone of log-concavity. As illustration, $C^2_6$ has 13 extreme vectors, which are

$$s_6 \quad s_4 s_2 \quad s_3 s_{2,1}$$
$$s_{5,1} \quad s_{3,1} s_2 \quad (s_{2,1})^2$$
$$s_{4,2} \quad s_2 s_2 \quad s_2(s_{1,2})^2$$
$$s_{3,2} \quad s_2 s_{2,1} \quad s_{2,1}^3$$
$$s_3 s_1$$
Let $P_k$ denote the partitions with $\leq k$ parts. A pair of partitions $(\lambda, \mu)$ in $P_2$ is said to be *interlaced* if it satisfies one of the following conditions:

1. $\lambda = (\lambda_1 \geq \lambda_2 > 0), \mu = (\mu_1 \geq \mu_2 > 0)$ with $\lambda_1 > \mu_1 \geq \lambda_2 > \mu_2$,
2. $\lambda = (\lambda_1 > \lambda_2 > 0), \mu = (\mu_1 > 0)$ with $\lambda_1 \geq \mu_1 \geq \lambda_2$,
3. $\lambda = (\lambda_1 > 0), \mu = (\mu_1 > 0)$.

If $(\lambda, \mu)$ is not interlaced, it is said to be *nested*. These definitions differ somewhat from what we might usually call nested and interlaced because of the inequalities and partitions with one part.

Suppose $A = \{\alpha^1, \alpha^2, \ldots\}$ is a collection of partitions in $P_2$ where $\sum_i |\alpha^i| = N$. We write $s_A := \prod_1 s_{\alpha^i}$. These $s_A$ are the generating vectors of the cone $C^2_N$.

We say $A$ is *nested* if all pairs $(\alpha^i, \alpha^j)$ in $A$ are nested.

**Theorem.** If $A$ is not nested, then $s_A$ is not extreme.

**Proof.** If $A$ is not nested, then at least one pair $(\lambda, \mu)$ in $A$ is interlaced and so satisfies one of the three interlacing conditions. Suppose that $\lambda = (\lambda_1 \geq \lambda_2 > 0), \mu = (\mu_1 \geq \mu_2 > 0)$ with $\lambda_1 > \mu_1 \geq \lambda_2 > \mu_2$. This implies $\lambda_1 \geq \mu_1 + 1$ and $\lambda_2 - 1 \geq \mu_2$. Therefore, by Jacobi-Trudi, $s_\lambda s_\mu = s_{(\lambda_1, \mu_2)} s_{(\mu_1, \lambda_2)} + s_{(\lambda_1, \mu_1+1)} s_{(\lambda_2-1, \mu_2)}$. The other two cases follow from similar identities. See [2] for details.

**Theorem.** If $A$ is nested and all the parts of $A$ are distinct, then $s_A$ is extreme.

The proof of this last theorem uses the Littlewood-Richardson rule in a non-trivial way and relies on Farkas' Lemma [1]. Farkas' Lemma states that a vector is extreme if and only if there is a hyperplane which separates it from all the other generating vectors. See [2] for details.

**OPAC-009.** Show that if $A$ is nested then $s_A$ is extreme.

Further information regarding this conjecture can be found in [2].

**OPAC-010.** Describe the extreme vectors of $C^3_N$.

**References:**


Matrix counting over finite fields

Submitted by Joel Brewster Lewis

Let \( r \leq m \leq n \) be positive integers, and \( q \) a prime power. Given a subset \( S \) of the discrete grid \( \{1, \ldots, m\} \times \{1, \ldots, n\} \), one may define the matrix count \( m_r(S; q) \) to be the number of rank-\( r \) matrices over the finite field of order \( q \) whose entries on \( S \) are equal to \( 0 \). This question concerns the properties of this matrix count as a function of \( q \).

A first basic property is that the integer \( m_r(S; q) \) is always divisible by \( (q - 1)^r \). (The idea of the proof is to consider the orbits formed when rescaling the rows by nonzero factors.) Consequently, it is convenient to define the reduced (or projective) matrix count \( M_r(S; q) := \frac{m_r(S; q)}{(q-1)^r} \).

One motivation for the study of the matrix counts comes from the classical enumerative combinatorics of rook theory: the rook number \( R_r(S) \) is the number of placements of \( r \) nonattacking rooks on \( \{1, \ldots, m\} \times \{1, \ldots, n\} \) so that none of them lies on \( S \). (Two rooks are attacking if they lie in the same row or same column, so this may equivalently be described as the number of \( m \times n \) partial permutation matrices whose support is disjoint from \( S \).) Then for any prime power \( q \) one has

\[
M_r(S; q) \equiv R_r(S) \pmod{q - 1},
\]

(see [4, Prop 5.1]) and so one may think of \( M_r(S; q) \) as a \( q \)-analogue of the rook number \( R_r(S) \).

Depending on the diagram \( S \), the reduced rook count may be more or less nice as a function of \( q \). When \( S \) is a Ferrers board (i.e., the diagram of an integer partition), Haglund [2, Thm 1] showed that the function \( M_r(S; q) \) is actually a polynomial in \( q \), with positive integer coefficients, and related to the \( q \)-rook number of Garsia and Remmel [1]. However, when \( S \) is arbitrary, the function \( M_r(S; q) \) need not be a polynomial function of \( q \) [7, Section 8.1], and in fact may be exceptionally complicated. It is natural to explore this boundary: which diagrams \( S \) give “nice” counting functions \( M_r(S; q) \)?

One natural way to extend Ferrers boards is to skew shapes, the set difference of two Ferrers boards. In fact, both are special cases of inversion diagrams of permutations (appearing in the literature under many names, including Rothe diagram): given a permutation \( w = w_1 w_2 \cdots w_n \) in one-line notation, the inversion diagram contains the box \((i, w_j)\)
whenever $i < j$ and $w_j < w_i$. Then any Ferrers board $S$ is (for some sufficiently large $n$) the inversion diagram of some $n$-permutation, and the $n$-permutations whose inversion diagrams are Ferrers boards are exactly those that avoid the permutation pattern 132, i.e., those for which there do not exist $i < j < k$ with $w_i < w_k < w_j$. Similarly, every skew shape is (after rearranging rows and columns; for some sufficiently large $n$) the inversion diagram of a 321-avoiding permutation. In [6, Cor. 4.6], it was shown that for any permutation $\mathcal{w}$ with inversion diagram $I_{\mathcal{w}}$, the matrix count $M_r(I_{\mathcal{w}}; q)$ is a polynomial function of $q$ with integer coefficients; but there exist permutations for which some of the coefficients are negative.

**OPAC-011.** Prove that if $\mathcal{w}$ is a 321-avoiding permutation, then the matrix count $M_r(I_{\mathcal{w}}; q)$ is a polynomial in $q$ with nonnegative integer coefficients.

This is essentially Conjecture 6.9 of [6]. It has been checked for all 321-avoiding permutations of size 14 or less.

One particular special case is worth mentioning. When $n$ is even, the permutation $\mathcal{w} = 214365 \cdots n(n-1)$ avoids 321; its diagram consists of exactly $n/2$ of the $n$ diagonal boxes in \{1, \ldots, $n$\} $\times$ \{1, \ldots, $n$\}. In this case, we have an explicit formula for $M_n(I_{\mathcal{w}}; q)$: define the standard $q$-number $[n]_q := 1 + q + \ldots + q^{n-1}$ and $q$-factorial $[n]_q! := [1]_q \cdot [2]_q \cdots [n]_q$, then one has $M_n(I_{\mathcal{w}}; q) = q^K \sum_{i=0}^{n} (-1)^i \binom{n}{i} [2n - i]_q!$ for some integer $K$ [6, Section 6.3].

**OPAC-012.** The sum $\sum_{i=0}^{n} (-1)^i \binom{n}{i} [2n - i]_q!$ is manifestly a polynomial with integer coefficients; prove that in fact the coefficients are nonnegative integers.

OPAC-012 is essentially Conjecture 6.8 of [6]. It is easy to verify on a computer for $n \leq 40$. Ideally, one would hope for a solution method that could be applied to other cases of OPAC-011, as well.

For more open questions along these lines, see [3] and [5].

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**References:**


Descents and cyclic descents

Submitted by Ron M. Adin and Yuval Roichman

The descent set of a permutation \( \pi = [\pi_1, \ldots, \pi_n] \) in the symmetric group \( S_n \) on \( [n] := \{1, 2, \ldots, n\} \) is

\[
\text{Des}(\pi) := \{1 \leq i \leq n - 1 : \pi_i > \pi_{i+1}\} \subseteq [n - 1]
\]

whereas its cyclic descent set is

\[
c\text{Des}(\pi) := \{1 \leq i \leq n : \pi_i > \pi_{i+1}\} \subseteq [n]
\]

with the convention \( \pi_{n+1} := \pi_1 \); see, e.g., [3, 4].

The descent set of a standard Young tableau (SYT) \( T \) is

\[
\text{Des}(T) := \{1 \leq i \leq n - 1 : i + 1 \text{ appears in a lower row of } T \text{ than } i\}.
\]

For a set \( I \subseteq [n - 1] \) let \( x^I := \prod_{i \in I} x_i \) and \( y^I := \prod_{i \in I} y_i \). The Robinson-Schensted correspondence implies

\[
(\ast) \quad \sum_{\pi \in S_n} x^{\text{Des}(\pi)} y^{\text{Des}(\pi^{-1})} = \sum_{\lambda \vdash n} \sum_{P, Q \in \text{SYT}(\lambda)} x^{\text{Des}(Q)} y^{\text{Des}(P)},
\]

where the first summation in the RHS is over all partitions of \( n \), and \( \text{SYT}(\lambda) \) denotes the set of all SYT of shape \( \lambda \).

**OPAC-013.** Find a cyclic analogue of Equation (\( \ast \)).

As a first step, note that Equation (\( \ast \)) implies

\[
(\ast\ast) \quad \sum_{\pi \in S_n} x^{\text{Des}(\pi)} = \sum_{\lambda \vdash n} \#\text{SYT}(\lambda) \sum_{T \in \text{SYT}(\lambda)} x^{\text{Des}(T)}.
\]

**Definition [1].** Let \( n \geq 2 \) and let \( T \) be any finite set equipped with a descent map \( \text{Des} : T \to 2^{[n-1]} \). Consider the cyclic shift \( \text{sh} : [n] \to [n] \) mapping \( i \) to \( i + 1 \) (mod \( n \)).
A cyclic extension of the descent map $\text{Des}$ is a pair $(\text{cDes}, p)$ where $\text{cDes} : \mathcal{T} \rightarrow 2^{[n]}$ is a map and $p : \mathcal{T} \rightarrow \mathcal{T}$ is a bijection, satisfying the following axioms: for all $T \in \mathcal{T}$,

- (extension) $\text{cDes}(T) \cap [n - 1] = \text{Des}(T)$
- (equivariance) $\text{cDes}(p(T)) = \text{sh}(\text{cDes}(T))$
- (non-Escher) $\emptyset \subsetneq \text{cDes}(T) \subsetneq [n]$.

For example, letting $\mathcal{T} = \mathfrak{S}_n$ be the symmetric group, the map $\text{cDes}(\pi)$ defined above and the rotation $p([\pi_1, \pi_2, \ldots, \pi_{n-1}, \pi_n]) := [\pi_n, \pi_1, \pi_2, \ldots, \pi_{n-1}]$ determine a cyclic extension of the map $\text{Des}(\pi)$ defined above.

A cyclic extension of the tableaux descent map $\text{Des}(T)$ defined above, for SYT of rectangular shapes, was introduced in [9]. In fact, this descent map on SYT$(\lambda/\mu)$ has a cyclic extension if and only if the skew shape $\lambda/\mu$ is not a connected ribbon [1, Theorem 1.1]; a constructive proof of this result was recently given in [7]. All cyclic extensions of $\text{Des}$ on SYT$(\lambda/\mu)$ share the same distribution of $\text{cDes}$.

The following cyclic analogue of (**) was proved in [1, Theorem 1.2]:

$$
(***) \sum_{\pi \in \mathfrak{S}_n} x^{\text{cDes}(\pi)} = \sum_{\lambda \vdash n, \text{non-hook}} \#\text{SYT}(\lambda) \sum_{T \in \text{SYT}(\lambda)} x^{\text{cDes}(T)} + \sum_{k=1}^{n-1} \binom{n-2}{k-1} \sum_{T \in \text{SYT}(1^k \oplus (n-k))} x^{\text{cDes}(T)}
$$

**OPAC-014.** Find a Robinson-Schensted-style bijective proof of Equation (***)

By a classical theorem of Gessel and Reutenauer [5, Theorem 2.1], there exists a collection of non-negative integers $\{m_{\lambda, \mu}\}_{\lambda, \mu \vdash n}$ such that for every conjugacy class $C_{\mu}$ of type $\mu$ in $\mathfrak{S}_n$

$$
(\dagger) \sum_{\pi \in C_{\mu}} x^{\text{Des}(\pi)} = \sum_{\lambda \vdash n} m_{\lambda, \mu} \sum_{T \in \text{SYT}(\lambda)} x^{\text{Des}(T)}.
$$

**OPAC-015.** Find a bijective proof of Equation (\dagger).

A bijective proof of a cyclic extension of Equation (\dagger), like the one given in [6, Theorem 6.2], is also desired.

Thrall [11] asked for a description of the coefficients of in Equation (\dagger); for recent discussions see, e.g., [8, 2, 10]. Particularly appealing is a combinatorial interpretation of $m_{\lambda, \mu}$ as the cardinality of a nice set of objects. This has been done in some special cases – for example, when $\lambda$ is a hook-shaped partition:
\[ m_{(n-k,1^k),\mu} = \# \{ \pi \in C_\mu : \text{Des}(\pi) = [k] \} \quad (0 \leq k \leq n-1). \]

**OPAC-016.** For which partitions \( \mu \vdash n \) is the sequence \( \{ m_{(n-k,1^k),\mu} \}_{k=0}^{n-1} \) unimodal?

It is known that this sequence is unimodal for \( \mu = (n) \), and conjecturally the same holds for all rectangular shapes \( \mu = (r^s) \); see [6].

Unlike the full symmetric group, when restricted to a general conjugacy class the definition of \( c\text{Des}(\pi) \) given above does not yield a cyclic extension of \( \text{Des} \). However, the following holds.

**Theorem [6, Theorem 1.4].** The descent map \( \text{Des} \) on a conjugacy class \( C_\mu \) of \( S_n \) has a cyclic extension \( (c\text{Des}, p) \) if and only if the partition \( \mu \) is not of the form \( (r^s) \) for a square-free \( r \).

The proof involves higher Lie characters and does not provide an explicit description of the extension.

**OPAC-017.** Find an explicit combinatorial description for the cyclic extension of \( \text{Des} \) on a conjugacy class of \( S_n \), whenever such an extension exists.

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**References:**


**Root polytope projections**

Submitted by **Sam Hopkins**

Let $\Phi$ be a crystallographic root system in an Euclidean vector space $V$ with inner product $\langle \cdot, \cdot \rangle$. For any subspace $U \subseteq V$ let $\pi_U : U \rightarrow V$ denote the orthogonal (with respect to $\langle \cdot, \cdot \rangle$) projection. We call a nonzero subspace $\{0\} \neq U \subseteq V$ a $\Phi$-subspace if $\Phi \cap U$ spans $U$. In this case $\Phi \cap U$ is a (crystallographic) root system in $U$.

Recall that the polytope $\text{ConvHull}(\Phi)$, which is the convex hull of all the roots, is called the root polytope of $\Phi$ (see, e.g., [1]). Let $\{0\} \neq U \subseteq V$ be a $\Phi$-subspace. Define $\kappa(\Phi, U)$ to be the minimal $\kappa \geq 1$ for which

$$\pi_U(\text{ConvHull}(\Phi)) \subseteq \kappa \cdot \text{ConvHull}(\Phi \cap U).$$

In other words, $\kappa(\Phi, U)$ is how much we need to dilate the root polytope of $\Phi \cap U$ by to contain the projection of the root polytope of $\Phi$.

**Example.** Let $V$ be $\mathbb{R}^4$ with its standard orthonormal basis $e_1, e_2, e_3, e_4$. Let

$$\Phi = \{ \pm(e_i - e_j), \pm(e_i + e_j) : 1 \leq i < j \leq 4 \},$$

i.e., $\Phi = D_4$. Let us use the notation

$$\alpha_1 := e_1 - e_2, \quad \alpha_2 := e_2 - e_3, \quad \alpha_3 := e_3 - e_4, \quad \alpha_4 := e_3 + e_4.$$

Let $U := \text{Span}_{\mathbb{R}}\{\alpha_1, \alpha_3, \alpha_4\}$. Note that $U \subseteq V$ is the subspace orthogonal to $\omega_2 := e_1 + e_2$. Thus for instance we can compute

$$\pi_U(\alpha_2) = \alpha_2 - \frac{\langle \alpha_2, \omega_2 \rangle}{\langle \omega_2, \omega_2 \rangle} \omega_2 = -\frac{1}{2}e_1 + \frac{1}{2}e_2 - e_3 = -\frac{1}{2}\alpha_1 - \frac{1}{2}\alpha_3 - \frac{1}{2}\alpha_4.$$

In fact, the projection

$$\pi_U(\Phi) = \{ \pm\frac{1}{2}\alpha_1 \pm \frac{1}{2}\alpha_3 \pm \frac{1}{2}\alpha_4, \pm\alpha_1, \pm\alpha_3, \pm\alpha_4 \}$$

consist of 14 points. On the other hand, it is easy to see that
\[ \Phi \cap U = \{ \pm \alpha, \pm \alpha_3, \pm \alpha_4 \}. \]

Thus \( \pi_U(\text{ConvHull}(\Phi)) \) is a rhombic dodecahedron, and \( \text{ConvHull}(\Phi \cap U) \) is an octahedron inscribed inside this rhombic dodecahedron. So \( \kappa(\Phi, U) = \frac{3}{2} \).

In [2, Lemma 2.10] it is shown that \( \kappa(\Phi, U) < 2 \) (and this exact value 2 turns out to be important for applications in that paper). However, the proof there unfortunately ultimately relies on a case-by-case analysis, leading to the following open problem:

**OPAC-018.** Prove in a uniform way (i.e., without relying on the classification of root systems) that \( \kappa(\Phi, U) < 2 \).

Let \( \kappa(\Phi) \) be the max of \( \kappa(\Phi, U) \) over all \( \Phi \)-subspaces \( U \). It turns out that \( \kappa(\Phi) \) can get arbitrarily close to 2. Indeed \( (2 - \kappa(\Phi)) \) is on the order of \( \text{rank}(\Phi)^{-1} \) (see [2, Table 8]). The root system which minimizes \( (2 - \kappa(\Phi)) \times \text{rank}(\Phi) \) is \( \Phi = E_8 \) for which this quantity is equal to \( \frac{8}{30} \).

**OPAC-019.** Give a root system-theoretic interpretation of \( \kappa(\Phi, U) \) or \( \kappa(\Phi) \) (e.g., in terms of other fundamental invariants like the Coxeter number, the degrees, the index of connection, et cetera).

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**References:**


The restriction problem

Submitted by Mike Zabrocki

A representation of $\text{GL}_n(\mathbb{C})$ is a homomorphism $\phi$ from $\text{GL}_n(\mathbb{C})$ to $\text{GL}_k(\mathbb{C})$. The value of $k$ is the dimension of the representation.

Up to isomorphism, there is one irreducible polynomial $\text{GL}_n(\mathbb{C})$ representation for each partition $\lambda$ with the length of $\lambda$ less or equal to $n$. The character of that irreducible representation is the Schur function $s_\lambda(x_1, x_2, \ldots, x_n)$ indexed by the partition $\lambda$. The dimension of that representation is the number of column strict tableaux of shape $\lambda$ with entries in $\{1, 2, \ldots, n\}$.

Since the permutation matrices are a natural subgroup of $\text{GL}_n(\mathbb{C})$, when an irreducible $\text{GL}_n(\mathbb{C})$ representation is restricted from $\text{GL}_n(\mathbb{C})$ to $S_n$ it decomposes as a direct sum of irreducible representations.

The restriction problem is the following:

$\text{OPAC-020}$. Find a combinatorial description of the decomposition of the irreducible $\text{GL}_n(\mathbb{C})$ module indexed by the partition $\lambda$ into symmetric group $S_n$-irreducibles.

This problem has a very long history, but generally very few people publish partial progress or failed attempts so there is very little written about it after the 1980’s beyond special cases.

To determine how a $\text{GL}_n(\mathbb{C})$ irreducible decomposes into $S_n$ irreducibles we can use the character $s_\lambda(x_1, x_2, \ldots, x_n)$ of the irreducible $\text{GL}_n(\mathbb{C})$ module and its evaluation at eigenvalues of permutation matrices $S_n \subseteq \text{GL}_n(\mathbb{C})$. Let $\gamma$ be a partition of $n$ and $(\zeta_\gamma, 1, \zeta_\gamma, 2, \ldots, \zeta_\gamma, n)$ be the eigenvalues of a permutation matrix of cycle structure $\gamma$ (up to reordering, this list only depends on the cycle structure).

If we evaluate the symmetric function $s_\lambda(x_1, x_2, \ldots, x_n)$ at the eigenvalues $(\zeta_\gamma, 1, \zeta_\gamma, 2, \ldots, \zeta_\gamma, n)$ this is the value of the $S_n$ character at a permutation of cycle structure $\gamma$.

Representation theory provides a formula for the multiplicity for a symmetric group irreducible indexed by $\mu$ (where $\mu$ is a partition of $n$ and the character of this irreducible is denoted $\chi^{(\mu)}$). It is equal to
\[ A_{\lambda,\mu} := \sum_{\gamma} s_\lambda(\zeta_{\gamma,1}, \zeta_{\gamma,2}, \ldots, \zeta_{\gamma,n}) \chi^\mu(\gamma) / z_\gamma \]

where the sum is over all partitions \( \gamma \) of \( \eta \).

Computing a few examples of this formula should indicate why it is not a particularly satisfactory answer beyond as a means of arriving at a numerical value. Littlewood [2, 9] showed in the 50’s that the multiplicity can be computed using the operation of plethysm:

\[ A_{\lambda,\mu} = \langle s_\lambda, s_\mu[1 + s_1 + s_2 + \cdots] \rangle \]

This is an advance in the problem, but recasts the solution of one problem in terms of another for which we don’t have a combinatorial formula.

I first became interested in this problem in the early 2000’s because, from time to time, I would encounter a module for which a formula for the \( Gl_n(\mathbb{C}) \) character was well known, but the symmetric group module structure was not. Then in 2016, Rosa Orellana and I [5] found a basis of the symmetric functions that are the the characters of the symmetric group as permutation matrices \( S_n \subseteq Gl_n(\mathbb{C}) \) in the same way that the Schur functions are characters of \( Gl_n(\mathbb{C}) \). That is, there is a basis \( \{ \tilde{s}_\mu \} \) (and one could take the following formula as a definition of this basis) such that for all \( n \) sufficiently large,

\[ s_\lambda = \sum_{\mu: |\mu| \leq |\lambda|} A_{\lambda, (n-|\mu|,\mu)} \tilde{s}_\mu. \]

Then, for \( \gamma \) a partition of \( \eta \), we have \( \tilde{s}_\mu(\zeta_{\gamma,1}, \zeta_{\gamma,2}, \ldots, \zeta_{\gamma,n}) = \chi^{(n-|\mu|,\mu)}(\gamma) \).

For each partition \( \lambda \), following symmetric function encodes all of the values of the symmetric group character of this representation:

\[ F_{\lambda,n} := \sum_{\gamma} s_\lambda(\zeta_{\gamma,1}, \zeta_{\gamma,2}, \ldots, \zeta_{\gamma,n}) \frac{p_\gamma}{z_\gamma} \]

where the sum is over all \( \gamma \) partitions of \( \eta \). An answer to the restriction problem would provide a Schur expansion of this expression as a symmetric function of degree \( \eta \). Note that if \( \ell(\lambda) > n \), then \( F_{\lambda,n} = 0 \).

Programs for computing data are easily accessible in Sage [7, 8] through the ring of symmetric functions. For instance, the following code:

```
sage: s = SymmetricFunctions(QQ).schur()
sage: s[3].character_to_frobenius_image(4)
```
computes the Schur expansion of \( F_{(3),4} \) by evaluating the character \( s_{(3)}(x_1, x_2, x_3, x_4) \) at the eigenvalues of permutation matrices and computing the Schur expansion of that expression.
In the case when $\lambda = (r)$, we have the following, which should be a special case of what the answer might look like in general:

**Proposition.** (Reformulation of [1]; see Exercise 7.73 of [10]; MacMahon’s Master Theorem [4] can be used to derive this.) The coefficient of the Schur function $s_\mu$ in $F(r)_n$ (where $\mu$ is a partition of $n$) is equal to the coefficient of $q^r$ in the Schur function evaluation $s_\mu(1, q, q^2, \ldots)$.

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**References:**


A localized version of Greene’s theorem

Submitted by Joel Brewster Lewis

Here is one collection of permutation statistics associated to a permutation \( w \) in the symmetric group \( S_n \) viewed as a sequence containing each element of \( \{1, \ldots, n\} \) exactly once: for any \( k \geq 0 \), let \( A_k \) be the maximum size of the disjoint union of \( k \) increasing subsequences of \( w \). For example, if \( w = 236145 \) then \( A_0 = 0 \), \( A_1 = 4 \) (witnessed uniquely by the subsequences \( 2345 \)), \( A_2 = 6 \) (witnessed uniquely by the pair of subsequences \( \{23, 6, 145\} \)), and \( A_k = 6 \) for all \( k > 2 \). Similarly, one can define a second collection \( D_k \) of permutation statistics by instead taking decreasing subsequences; with \( w = 236145 \) one has \( D_0 = 0 \), \( D_1 = 2 \), \( D_2 = 4 \), \( D_3 = 5 \), and \( D_k = 6 \) for all \( k > 3 \). The following paraphrase of a famous theorem of Greene explains how these sequences are related to each other.

**Theorem** (Greene [1, Thm. 3.1]). Let \( w \) be a permutation in \( S_n \) with \( A_k \), \( D_k \) as above. For \( k \geq 1 \), let \( \lambda_k := A_k - A_{k-1} \) and \( \mu_k := D_k - D_{k-1} \). Then the sequences \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \) are weakly decreasing sequences of nonnegative integers with sum \( n \) (that is, they are integer partitions of \( n \)); in fact, they are conjugate partitions, in the sense that \( \mu_i \) is equal to the number of parts of \( \lambda \) of size larger than or equal to \( i \), and vice-versa.

Of course the excitement of the theorem is not just that \( \lambda \) and \( \mu \) are any pair of conjugate partitions, but that they are a particularly meaningful pair: \( \lambda \) is exactly the shape of the standard Young tableau associated to \( w \) by the Robinson-Schensted correspondence.

We now describe a “localized” version of the quantities \( A_k \) and \( D_k \).

An ascent in a sequence \( u = (u_1, u_2, \ldots) \) is an index \( i \) such that \( u_i < u_{i+1} \). Let \( \text{asc}(u) \) denote the number of ascents of \( u \), and let \( \text{asc}^*(u) := \begin{cases} 0 & \text{if } u \text{ is empty}, \\ 1 + \text{asc}(u) & \text{otherwise}. \end{cases} \)

Given a permutation \( w \) in the symmetric group \( S_n \) define

\[
A'_k := \max_{u_1, \ldots, u_k} (\text{asc}^*(u_1) + \cdots + \text{asc}^*(u_k))
\]

where the maximum is taken over disjoint subsequences \( u_i \) of \( w \). For example, with \( w = 63417285 \) one has \( A'_0 = 0 \), \( A'_1 = \text{asc}^*(w) = 4 \), \( A'_2 = 7 \) (one can take subsequences
6125 and 3478), and $A'_k = 8$ for all $k \geq 3$ (one can take subsequences $67, 348, 125$ among many other options). On the other hand, for a sequence $u$, define $d(u)$ to be the longest decreasing subsequence of $u$, and define

$$D'_k = \max_{w = u_1 \cdots u_k} (d(u_1) + \cdots + d(u_k))$$

where the maximum is taken over ways of writing $w$ as a concatenation $u_1 \cdots u_k$ of subsequences (now obliged to be consecutive). For example, with $w = 63417285$, one has $D'_0 = 0$, $D'_1 = d(w) = 3$, $D'_2 = 5$ (witnessed by $6341 \mid 7285$, among other divisions), $D'_3 = 7$ (witnessed by $6341 \mid 72 \mid 85$) and $D'_k = 8$ for $k \geq 4$ (witnessed by $63 \mid 41 \mid 72 \mid 85$).

The following theorem shows that these localized versions are again closely related.

\textbf{Theorem} (Lewis–Lyu–Pylyavskyy–Sen [3, Lem. 2.1]). Let $w$ be a permutation in $\mathfrak{S}_n$ with $A'_k \leq D'_k$ as above. For $k \geq 1$, let $\lambda'_k := A'_k - A'_{k-1}$ and $\mu'_k := D'_k - D'_{k-1}$. Then the sequences $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ and $\mu' = (\mu'_1, \mu'_2, \ldots)$ are weakly decreasing sequences of nonnegative integers with sum $n$ (that is, they are integer partitions of $n$); in fact, they are conjugate partitions, in the sense that $\mu'_i$ is equal to the number of parts of $\lambda'$ of size larger than or equal to $i$, and vice-versa.

Again, the excitement of the theorem has something to do with the specific meaning of the partition. In this case, $\mu'$ is the soliton partition describing the long-term behavior of a multicolor box-ball system (BBS) initialized with one ball in each color $\{1, \ldots, n\}$, arranged according to $w$. Here the BBS is a dynamical system consisting of balls in an infinite strip; balls take turns jumping to the first available cell, beginning with the largest-numbered ball. For example, using $|$s to denote empty cells and beginning with the initial configuration $634172850000 \cdots$, one BBS move (in which all balls jump once, starting with ball 8 and ending with ball 1) results in the new position $0030617485200 \cdots$. A second move produces the configuration $0003006174085200 \cdots$, and a third move produces the configuration $0000300061740085200 \cdots$. At every subsequent time-step, the three balls 852 advance three steps to the right, the two pairs 74 and 61 advance two steps to the right, and the singleton 3 advances one step to the right. These unchanging sequences are the solitons, and the soliton partition $(3, 2, 2, 1)$ records their length. Not only does this partition equal $\mu$, but the proof of the above theorem uses the box-ball dynamics in an essential way, by suitably interpreting the statistics $A'_k$ and $D'_k$ for more general BBS configurations, showing that they are preserved under a step of the system, and showing that they give conjugate partitions once the system has decomposed into solitons.

\textbf{OPAC-021}. Give a direct proof of the above theorem concerning $A'_k$ and $D'_k$ that stays inside the realm of permutation combinatorics (i.e., not using the full machinery of the box-ball system).

(It is not difficult but also not trivial to prove that $\lambda'_1$ is the number of positive parts of $\mu'$ and vice-versa. It is also not too hard to show an inequality between $\lambda'$ and the conjugate of
Greene’s invariants $A_k, D_k$ may be defined more generally for any finite poset $P$, by considering maximum collections of chains and antichains [2]. These are called the Greene-Kleitman invariants of the poset. (One recovers the permutation case by considering a permutation as a certain poset of dimension 2.) It is natural to ask the same for the localized versions.

**OPAC-022.** Is there a “localized” version of the Greene-Kleitman invariants that specializes to the quantities $A'_k, D'_k$ in some case naturally associated to permutations?

**References:**


Descent sets for tensor powers

Submitted by Bruce W. Westbury

Let $T$ be a standard tableau of size $n$. The descent set of $T$, $\text{Des}(T)$, is the subset of $[n - 1] := \{1, 2, \ldots, n - 1\}$ consisting of those $i$ for which $i + 1$ appears in a lower row than $i$.

For each subset $D \subseteq [n - 1]$ we have a fundamental quasisymmetric function $F_D$.
(See [5, Ch. 7] for background on symmetric and quasisymmetric functions.) A basic fact is that the combinatorics of descents gives the quasisymmetric expansions of the Schur functions. Let $s_\lambda$ be the Schur function associated to the partition $\lambda$. Then, for all partitions $\lambda$, we have the expansion

$$s_\lambda = \sum_{T : \text{sh}(T) = \lambda} F_{\text{Des}(T)}.$$  

Let $V$ be a highest weight representation of a reductive algebraic group or Lie algebra. For each highest weight $\lambda$ we have an irreducible representation $V(\lambda)$. Then, for each $n \geq 0$, we have the decomposition

$$\otimes^n V = \sum_{\lambda} U(n, \lambda) \otimes V(\lambda)$$

where $U(n, \lambda)$ is the space of highest weight tensors of weight $\lambda$.

Each isotypic subspace, $U(n, \lambda)$, has a natural action of the symmetric group $\mathfrak{S}_n$ and hence a Frobenius character, $\text{ch}(n, \lambda)$. The problem is to find the quasisymmetric expansion of this symmetric function.

Let $C$ be the crystal of $V$. Then, for each $n \geq 0$, we have the decomposition

$$\otimes^n C = \sum_{\lambda} W(n, \lambda) \times C$$

where $W(n, \lambda)$ is the set of highest weight words of weight $\lambda$. 
A descent set is a function \( \text{Des} : W(n, \lambda) \to 2^{[n-1]} \) such that

\[
\text{ch}(n, \lambda) = \sum_{T \in \text{Des}(T)} F_{\text{Des}(T)}
\]

It is clear that descent sets in this sense exist since the quasisymmetric expansion of \( \text{ch}(n, \lambda) \) corresponds to a multiset of subsets of \([n-1]\) whose cardinality is the cardinality of \( W(n, \lambda) \). However the problem is to give a construction.

**OPAC-023.** Give an explicit construction of descent sets for various representations \( V \).

Here is a classical example: take \( V \) to be the vector representation of \( GL(V) \). Then, by Schur-Weyl duality, we can identify \( W(n, \lambda) \) with the set of standard tableaux of shape \( \lambda \), and the aforementioned combinatorial definition of the descent set of a standard tableau gives us a descent set in this sense.

The current situation is that descent sets are only known for the vector representations of classical groups; that is, for the vector representation of a general linear group (as just explained), for the vector representation of a symplectic group \([1]\), and for the vector representation of an orthogonal group \([2, 3]\).

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**References:**


