Ehrhart polynomial of a polytope plus dilating zonotope

University of Minnesota Combinatorics Seminar

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September 14th, 2018

Joint work with Alex Postnikov

Interval-firing processes

In earlier work with Pavel Galashin, Thomas McConville, and Alex Postnikov we introduced *interval-firing processes*, certain digraphs on $\mathbb{Z}^N$.

For $k \in \mathbb{Z}_{\geq 0}$, the *symmetric interval-firing process* has edges

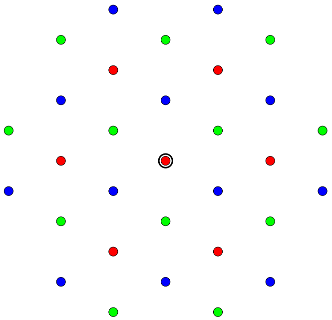
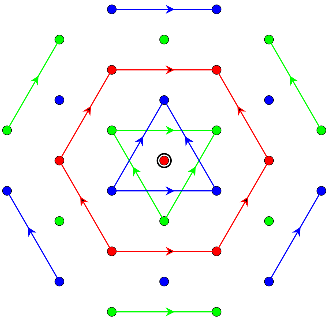
$$\vec{a} \xrightarrow[\mathrm{sym},k]{} \vec{a}+(e_i-e_j) : \vec{a} \in \mathbb{Z}^N, 1 \leq i < j \leq N, \langle \vec{a}, e_i-e_j \rangle + 1 \in \{-k, \ldots, k\};$$

and the *truncated interval-firing process* has edges

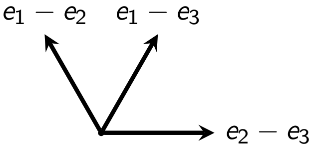
$$\vec{a} \xrightarrow[\mathrm{tr},k]{} \vec{a} + (e_i - e_j) : \text{—"—}, \langle \vec{a}, e_i - e_j \rangle + 1 \in \{-k+1, \ldots, k\}.$$

To get a sense of why these digraphs are in any way interesting, it's best to look at some pictures...

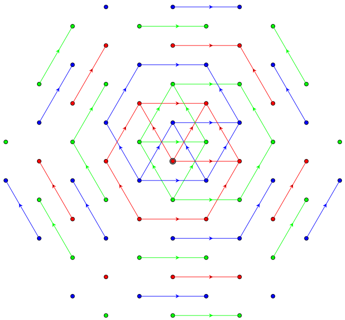
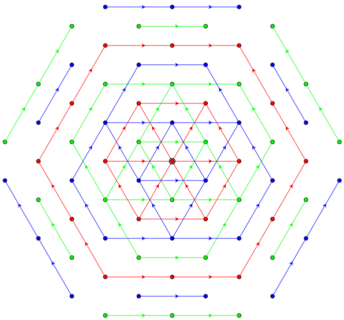
Pictures of interval-firing for $N = 3$

k	truncated	symmetric
0		

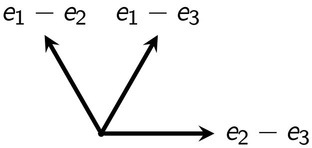
Projection of $\mathbb{Z}^3$ to plane orthogonal to $(1, 1, 1)$; the color represents sum modulo 3 of coordinates.



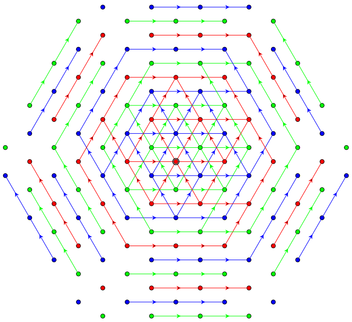
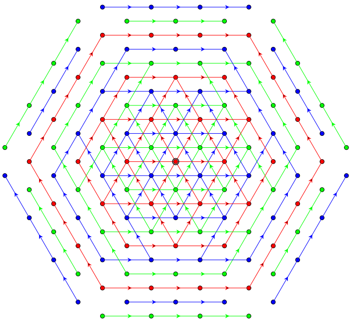
Pictures of interval-firing for $N = 3$

k	truncated	symmetric
1		

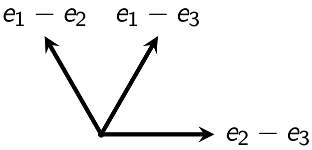
Projection of $\mathbb{Z}^3$ to plane orthogonal to $(1, 1, 1)$; the color represents sum modulo 3 of coordinates.



Pictures of interval-firing for $N = 3$

k	truncated	symmetric
2		

Projection of $\mathbb{Z}^3$ to plane orthogonal to $(1,1,1)$; the color represents sum modulo 3 of coordinates.



Confluence of interval-firing

One of the main results we proved about interval-firing is the following *confluence* result:

Theorem (GHMP)

For any $k \in \mathbb{Z}_{\geq 0}$, and for either the symmetric or truncated interval-firing process, each connected component of the digraph has a unique sink.

The sinks give us a way to index these components.

Lemma (GHMP)

(Some of) these sinks are $\vec{a} + k\vec{\rho}$, where $\vec{a} = (a_1 \geq a_2 \geq \cdots \geq a_N) \in \mathbb{Z}^N$ is any weakly decreasing vector and $\vec{\rho} := (N-1, N-2, \ldots, 0)$.

Next goal: understand these components in more detail.

Ehrhart-like polynomials

From the above pictures, it looks like for any $\vec{a} = (a_1 \geq \cdots \geq a_N) \in \mathbb{Z}^N$, the connected component with sink $\vec{a} + k\vec{\rho}$ gets "dilated" as k grows.

In analogy with the Ehrhart polynomial of a polytope, define

$$L^{\mathrm{sym}}_{\vec{a}}(k) := \#\text{ of points in }\xrightarrow[\mathrm{sym},k]{}\text{-component containing }\vec{a} + k\vec{\rho};$$

$$L^{\mathrm{tr}}_{\vec{a}}(k) := \#\text{ of points in }\xrightarrow[\mathrm{tr},k]{}\text{-component containing }\vec{a} + k\vec{\rho}.$$

Theorem (GHMP)

Both $L^{\mathrm{sym}}_{\vec{a}}(k)$ and $L^{\mathrm{tr}}_{\vec{a}}(k)$ are polynomials in k.

Conjecture (GHMP)

These polynomials have nonnegative integer coefficients.

We prove this for $L^{\mathrm{sym}}_{\vec{a}}(k)$ (and even give a formula- thanks, Dennis!).

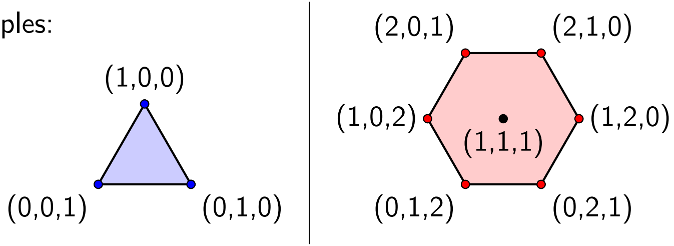
Permutohedra

In the pictures of the symmetric interval-firing process, you may have seen some highly symmetric, polytopal shapes. These are permutohedra.

For $\vec{a} \in \mathbb{Z}^N$, the *permutohedron of $\vec{a}$* is

$$\Pi(\vec{a}) := \mathrm{ConvexHull}\,\{(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(N)})\colon \sigma \in S_N\}.$$

Examples:



Important: the *regular permutohedron* $\Pi(\vec{\rho})$ is a zonotope:

$$\Pi(\vec{\rho}) = \sum_{1 \leq i < j \leq N} [e_i, e_j].$$

Symmetric components as differences of permutohedra

Let $\vec{a} = (a_1 \geq a_2 \geq \cdots \geq a_N) \in \mathbb{Z}^N$ be s.t. $a_i - a_{i+1} \in \{0, 1\}$ for all i. Understanding $\xrightarrow[\mathrm{sym},k]{}$-components containing $\vec{a} + k\vec{\rho}$ for $\vec{a}$ of this form is enough to understand all components (others are "lower-dimensional").

Lemma (GHMP)

For $\vec{a}$ of this form, the $\xrightarrow[\mathrm{sym},k]{}$-component containing $\vec{a} + k\vec{\rho}$ is

$$(\Pi(\vec{a} + k\vec{\rho}) \cap \mathbb{Z}^N) \setminus \bigcup_{\substack{\vec{b}=(b_1 \geq \cdots \geq b_N) \in \mathbb{Z}^N, \\ \vec{b} \leq \vec{a}}} (\Pi(\vec{b} + k\vec{\rho}) \cap \mathbb{Z}^N),$$

where $\vec{b} \leq \vec{a}$ if $\vec{a} = \vec{b} + \sum_{i=1}^{N-1} c_i(e_i - e_{i+1})$, $c_i \in \mathbb{Z}_{\geq 0}$ (dominance order).

This motivates us to study the lattice points in $\Pi(\vec{a} + k\vec{\rho}) = \Pi(\vec{a}) + k\Pi(\vec{\rho})$, or more generally the lattice points in a polytope plus dilating zonotope.

Ehrhart theory of lattice zonotopes

A *zonotope* is a Minkowski sum of line segments. Let $v_1, \ldots, v_m \in \mathbb{Z}^N$ be lattice vectors and $\mathcal{Z} := \sum_{i=1}^{m} [0, v_i]$ the corresponding lattice zonotope.

Theorem (Stanley, 1980)

$$\#(k\mathcal{Z} \cap \mathbb{Z}^N) = \sum_{\substack{X \subseteq \{v_1, \ldots, v_m\} \\ \textit{lin. ind.}}} \mathrm{rVol}(X) \cdot k^{\#X},$$

where rVol(X) is the gcd of the maximal minors of the matrix whose columns are the elements of X.

Corollary (Stanley)

$$\#(k\Pi(\vec{\rho}) \cap \mathbb{Z}^N) = \sum_{F \textit{ labeled forest on } N \textit{ vertices}} k^{\# \textit{edges in } F}.$$

(For corollary: we use total unimodularity of $\{e_i - e_j\}$.)

Lattice points in a polytope plus dilating zonotope

Theorem (HP)

Let $\mathcal{P}$ be any (lattice) polytope in $\mathbb{R}^N$ and $\mathcal{Z}$ as before. Then,

$$\#((k\mathcal{Z}+\mathcal{P})\cap\mathbb{Z}^N) = \sum_{\substack{X\subseteq\{v_1,\ldots,v_m\},\\ \textit{lin. ind.}}} \#(\mathrm{quot}_X(\mathcal{P})\cap\mathrm{quot}_X(\mathbb{Z}^N))\cdot\mathrm{rVol}(X)\cdot k^{\#X},$$

where $\mathrm{quot}_X\colon \mathbb{R}^N \twoheadrightarrow \mathbb{R}^N/\mathrm{Span}_{\mathbb{R}}(X)$ *is the canonical quotient map.*

The proof of this theorem is quite easy (follows from "multi-paramter" version of Ehrhart polynomials due to McMullen, 1977).

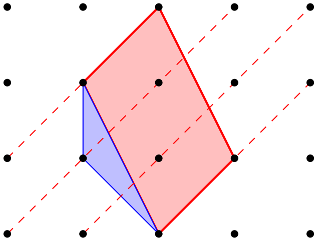
Quotients might not be so nice in general

The formula on the previous slide requires us to check every rational point of $\mathcal{P}$ because in general, $\mathrm{quot}_X(\mathcal{P} \cap \mathbb{Z}^N) \neq \mathrm{quot}_X(\mathcal{P}) \cap \mathrm{quot}_X(\mathbb{Z}^N)$:

$$\mathcal{P} = \mathrm{ConvexHull}\{(1,0),(0,1),(0,2)\}$$
$$X = \{(1,1)\}$$

$$\#\mathrm{quot}_X(\mathcal{P} \cap \mathbb{Z}^2) = 3$$
$$\#\mathrm{quot}_X(\mathcal{P}) \cap \mathrm{quot}_X(\mathbb{Z}^2) = 4$$

$$\#((\mathcal{P} + k[0,(1,1)]) \cap \mathbb{Z}^2) = 3 + 4k$$



Because of this, formula is not ideal from a combinatorial perspective.

Quotients are nice for permutohedra

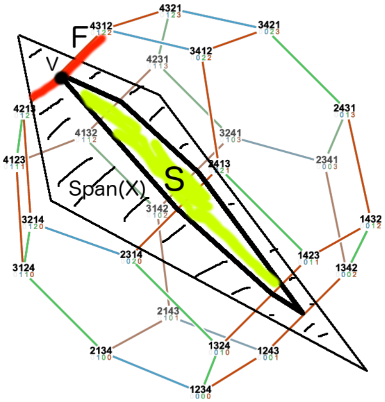
But for permtuohedra, bad behavior depicted on last slide does not happen:

Lemma (HP)

For $\vec{a} \in \mathbb{Z}^N$ *and* $X \subseteq \{e_i - e_j\}$,

$$\mathrm{quot}_X(\Pi(\vec{a}) \cap \mathbb{Z}^N) = \mathrm{quot}_X(\Pi(\vec{a})) \cap \mathrm{quot}_X(\mathbb{Z}^N).$$

Proof that quotients are nice for permutohedra



Let $\vec{b} \in \mathbb{Z}^N$ be such that $\mathcal{S} \neq \varnothing$, where $\mathcal{S} := \Pi(\vec{a}) \cap (\mathrm{Span}_{\mathbb{R}}(X) + \vec{b})$ ($\mathcal{S}$ = "slice"). We want to show that $\mathcal{S}$ contains an integer point. Let v be a vertex of $\mathcal{S}$. Let F be a face of $\Pi(\vec{a})$ of min. dimension containing v; by symmetry can assume $\vec{a} \in F$. Choose $\alpha_1, \ldots, \alpha_m \subseteq \{e_i - e_j\}$ spanning F; choose $\beta_{m+1}, \ldots, \beta_{N-1} \subseteq \{e_i - e_j\}$ spanning $\mathrm{Span}_{\mathbb{R}}(X)$.

Key point: by total unimodularity, $\mathrm{Span}_{\mathbb{Z}}\{\alpha_i, \beta_j\} = \mathrm{Span}_{\mathbb{Z}}\{e_i - e_j\}$. So $\vec{b} = \vec{a} + \sum_i c_i\alpha_i + \sum_j d_j\beta_j$ for $c_i, d_j \in \mathbb{Z}$. But $v = \vec{a} + \sum_i c_i\alpha_i$, so the vertex v *is* an integer point!

Permutohedron plus dilating regular permutohedron

Corollary (HP)

For $\vec{a} = (a_1 \geq \cdots \geq a_N) \in \mathbb{Z}^N$,

$$\#(\Pi(\vec{a} + k\vec{\rho}) \cap \mathbb{Z}^N) = \sum_{\substack{X \subseteq \{e_i - e_j\} \\ \textit{lin. ind.}}} \#(\mathrm{quot}_X(\Pi(\vec{a}) \cap \mathbb{Z}^N)) \cdot k^{\#X}$$

Not necessarily easy to give a formula for $\#(\mathrm{quot}_X(\Pi(\vec{a}) \cap \mathbb{Z}^N))$; but we can when $\vec{a} = (1, 1, \ldots, 1, 0, 0, \ldots, 0)$ ("minuscule weight"). For instance, with $f_\lambda := \prod_{i=1}^{\ell(\lambda)} \lambda_i^{\lambda_i - 2}$, we have:

$$\#(\Pi((1, 0, \ldots, 0) + k\vec{\rho}) \cap \mathbb{Z}^N) = \sum_{\lambda \vdash N} \ell(\lambda) \cdot f_\lambda \cdot k^{N - \ell(\lambda)};$$

$$\#(\Pi((1, 1, 0, \ldots, 0) + k\vec{\rho}) \cap \mathbb{Z}^N) = \sum_{\lambda \vdash N} \left(\binom{\lambda'_1}{2} + \lambda'_2 \right) \cdot f_\lambda \cdot k^{N - \ell(\lambda)}.$$

W-permutohedra

With the formula for $\#(\Pi(\vec{a}+k\vec{\rho})\cap\mathbb{Z}^N)$, together with the description of components as differences of permutohedra, we can use inclusion-exclusion on dominance order to get a positive formula for $L^{\mathrm{sym}}_{\vec{a}}(k)$.

But actually it is easiest to state this formula for general root systems.

Let Φ be a crystallographic *root system*: finite subset of Euclidean space V closed under reflection orthogonal to any *root* $\alpha\in\Phi$. $Q:=\mathrm{Span}_{\mathbb{Z}}(\Phi)$ is the *root lattice* and $P:=\{v\in V\colon \langle v,\alpha^\vee\rangle\in\mathbb{Z}\text{ for all }\alpha\in\Phi\}$ is the *weight lattice*. W is the *Weyl group*, and $\rho=\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ is the *Weyl vector*.

In our previous set up: $\Phi=\{e_i-e_j\}$, $Q=\{\vec{v}\in\mathbb{Z}^N\colon \vec{v}\perp(1,1,\ldots,1)\}$, $P=\mathbb{Z}^N$, $W=S_N$, $\rho=\vec{\rho}$ (this is "Type A_{N-1}").

Can define (W-)permutohedra exactly analogously: for $\lambda\in P$,

$$\Pi(\lambda):=\mathrm{ConvexHull}\,W(\lambda);\qquad \Pi^Q(\lambda):=\Pi(\lambda)\cap(Q+\lambda).$$

Quotients are nice for W-permutohedra

Lemma (HP)

For $\lambda \in P$ and $X \subseteq \Phi$, $\mathrm{quot}_X(\Pi^Q(\lambda)) = \mathrm{quot}_X(\Pi(\lambda)) \cap \mathrm{quot}_X(Q+\lambda)$.

The proof of this for general Φ is much more involved than in Type A, because we no longer have total unimodularity. We reduce the proof to the following "projection-dilation" property of root systems:

Lemma (HP)

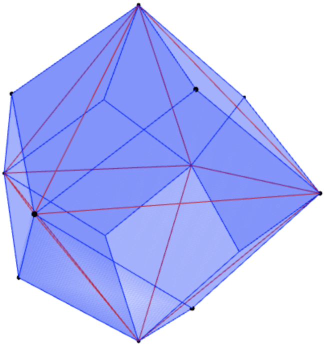
Let $\{0\} \neq U \subseteq V$ be any nonzero subspace spanned by a subset of Φ. Then there is some $1 \leq \kappa < 2$ such that

$$\pi_U(\mathrm{ConvexHull}(\Phi)) \subseteq \kappa \cdot \mathrm{ConvexHull}(\Phi \cap U),$$

where $\pi_U \colon V \to U$ is orthogonal projection.

But **we have no uniform proof of this last lemma!!!**

Projection-dilation property example: D_4



Let $\Phi = D_4$, and U the maximal parabolic subspace corresponding to the trivalent node in the Dynkin diagram. Then $\mathrm{ConvexHull}(\Phi \cap U)$ is an octahedron, while $\pi_U(\mathrm{ConvexHull}(\Phi))$ is a rhombic dodecahedron circumscribing it. Projection-dilation constant is $\kappa = \frac{3}{2}$.

Positive formula for symmetric Ehrhart-like polynomials

Corollary (HP)

For a dominant weight $\lambda \in P_{\geq 0}$, we have

$$\#(\Pi^Q(\lambda + k\rho)) = \sum_{\substack{X \subseteq \Phi \\ \textit{lin. ind.}}} \#(\mathrm{quot}_X(\Pi^Q(\lambda))) \cdot \mathrm{rVol}_Q(X) \cdot k^{\#X}.$$

By inclusion-exclusion on root order we get:

Theorem (HP)

Let $\lambda \in P$ be such that $\langle \lambda, \alpha_i^\vee \rangle \in \{0,1\}$ for all simple roots α_i. Then

$$L_\lambda^{\mathrm{sym}}(k) = \sum_{\substack{X \subseteq \Phi^+, \\ \textit{lin. ind.}}} \# \left\{ \mu \in W(\lambda) \colon \begin{array}{c} \langle \mu, \alpha^\vee \rangle \in \{0,1\} \textit{ for} \\ \textit{all } \alpha \in \Phi^+ \cap \mathrm{Span}_{\mathbb{R}}(X) \end{array} \right\} \cdot \mathrm{rVol}(X) \cdot k^{\#X}.$$

Truncated Ehrhart-like polynomials?

GHMP showed that for $\lambda \in P$ with $\langle \lambda, \alpha_i^\vee \rangle \in \{0,1\}$ for all simple roots α_i,

$$L_\lambda^{\mathrm{sym}}(k) = \sum_{\mu \in W(\lambda)} L_\mu^{\mathrm{tr}}(k).$$

Thus our symmetric formula very naturally suggests:

Conjecture

Let $\lambda \in P$. Then

$$L_\lambda^{\mathrm{tr}}(k) = \sum_X \mathrm{rVol}_Q(X)\, k^{\#X},$$

where the sum is over all $X \subseteq \Phi^+$ such that:

- *X is linearly independent;*
- *$\langle \lambda, \alpha^\vee \rangle \in \{0,1\}$ for all $\alpha \in \Phi^+ \cap \mathrm{Span}_{\mathbb{R}}(X)$.*

However, the above conjecture turns out to be **false in general**.
It fails for $\Phi = G_2, C_3, C_4, D_4$. But it may hold for Type A and B?

Thank you!

References:

- Galashin, Hopkins, McConville, Postnikov. "Root system chip-firing I: Interval-firing." `arXiv:1708.04850`. Forthcoming, *Mathematische Zeitschrift*.
- Hopkins, Postnikov. "A positive formula for the Ehrhart-like polynomials from root system chip-firing." `arXiv:1803.08472`