RESEARCH STATEMENT

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Broadly speaking, I study algebraic combinatorics, a field which has strong connections with commutative algebra, algebraic geometry, and representation theory.

Recently I have been especially involved in the growing subfield of *dynamical algebraic combinatorics*. Dynamical algebraic combinatorics investigates the dynamical properties of various natural operators acting on objects from algebraic combinatorics. Usually these objects come from finite *posets* (partially ordered sets), and the operators are algorithmically defined bijections. Dynamical algebraic combinatorics has intriguing ties with:

- root system and Coxeter group combinatorics,
- the representation theory of Lie algebras,
- cluster algebra dynamics and integrable systems more generally,
- quiver representations and the representation theory of associative algebras.

1. Current research

Over the past couple years, I have developed the following powerful heuristic:

(*) posets with good dynamical behavior = posets with order polynomial product formulas.

The *order polynomial* is a certain enumerative invariant of the poset. Meanwhile, "good dynamical behavior" means good behavior (e.g., small, predictable order and regular orbit structure) of *promotion of linear extensions* and *rowmotion of P-partitions*. Posets with order polynomial product formulas and posets with good dynamical behavior are both quite rare; in either case there are only a handful of known families. I have used the heuristic (*) to discover the first new examples of such families of posets in many years. Moreover, this heuristic has also pointed the way to interesting algebra underlying the observed combinatorial phenomena. I discuss this heuristic in detail in [19]. I will now briefly explain the recent successes I've had applying it.

1.1. **Order polynomial product formulas.** For a poset P, a *P-partition of height m* is an order-reversing map $P \to \{0, 1, \ldots, m\}$; we denote the set of such P-partitions by $\mathcal{PP}^m(P)$. The *order polynomial* $\Omega_P(m)$ of P is the polynomial in m for which $\Omega_P(m) = \#\mathcal{PP}^m(P)$. The order polynomial was introduced by Richard Stanley in his PhD thesis [40]. It is the same as the *Ehrhart polynomial* of the *order polytope* of P, and hence the study of order polynomials is closely related to polytopal geometry. In particular, the leading coefficient of $\Omega_P(m)$ is $1/\#P!$ times the number of *linear extensions* (total orderings of the elements extending the partial order) of P.

The number of linear extensions is perhaps the single most important numerical invariant associated to a finite poset. While it is known that computing this quantity is hard in general, there are many significant families of poset with nice formulas counting their linear extensions. For example, the famous *Hook Length Formula* gives a product formula for the number of linear extensions of posets of *Young diagram shape*.

Date: September 8, 2020.

5	3	3	2	2	1
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$\Leftrightarrow$

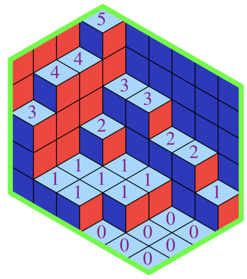


FIGURE 1. The correspondence between plane partitions and lozenge tilings.

Posets with product formulas for their entire order polynomial are comparatively much rarer. The first examples were provided by MacMahon in his investigation of *plane partitions*, which are the special case of P-partitions for $P = [a] \times [b]$ a *rectangle poset*, i.e., a product of two chains. MacMahon's celebrated product formula [29] for the size generating function of plane partitions is

$$\sum_{\pi \in \mathcal{PP}^m([a]\times[b])} q^{|\pi|} = \prod_{i=1}^{a} \prod_{j=1}^{b} \frac{(1-q^{i+j+m-1})}{(1-q^{i+j-1})}, \tag{1}$$

where $|\pi| := \sum_{p \in P} \pi(p)$ is the *size* of a P-partition. MacMahon's formula is nowadays recognized as one of the most elegant in algebraic and enumerative combinatorics. By setting $q := 1$ it gives a product formula for the order polynomial of the rectangle poset.

There are several alternate ways of viewing the plane partitions in $\mathcal{PP}^m([a]\times[b])$. They have a three-dimensional representation as a stacking of unit cubes in an $a \times b \times m$ box. In turn, this 3D picture can be thought of as a 2D picture, which shows that such plane partitions also correspond to *lozenge tilings* of a hexagonal region of the triangular lattice. See Figure 1 for this correspondence. Lozenge tilings are a special case of *dimer coverings* of a planar graph, and in this way plane partitions are related to exactly solvable models in statistical mechanics. Plane partitions also have strong ties with the representation theory of classical Lie algebras, which I'll say more about below.

P-partitions for posets P of other Young diagram shapes and *shifted shapes* beyond rectangles have also received significant attention because of their connection with the *symmetry classes* of plane partitions in a box. For instance, consider *transposition* $\mathrm{Tr}\colon \mathcal{PP}^m([n]\times[n]) \to \mathcal{PP}^m([n]\times[n])$. The set of *symmetric* (Tr-invariant) plane partitions in $\mathcal{PP}^m([n]\times[n])$ is evidently in bijection with $\mathcal{PP}^m(P)$ for P of *shifted staircase shape*. In 1898 MacMahon conjectured a product formula for the size generating function of symmetric plane partitions, and a little more than 75 year later Andrews [2] and Macdonald [28] independently proved MacMahon's conjecture. Thus, posets of shifted staircase shape have order polynomial product formulas. In the 1980s Proctor similarly obtained product formulas for posets of *staircase shape* [32] and of *shifted trapezoid shape* [33].

Very recently, Tri Lai and I obtained the first new example of a family of posets with order polynomial product formulas since the 80s:

Theorem 1 (Hopkins–Lai [20])**.** *For P a poset of* shifted double staircase shape *$(n, n-1, \ldots, 1) + (k, k-1, \ldots, 1)$ with $0 \le k \le n$, the order polynomial of P is*

$$\Omega_P(m) = \prod_{1 \le i \le j \le n} \frac{m+i+j-1}{i+j-1} \cdot \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j}$$

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline\end{array} \xrightarrow{\text{delete 1}} \begin{array}{|c|c|c|}\hline \bullet&2&4\\\hline 3&5&6\\\hline\end{array} \xrightarrow{\text{slide}} \begin{array}{|c|c|c|}\hline 2&4&6\\\hline 3&5&\bullet\\\hline\end{array} \xrightarrow{\text{decrement}} \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&\bullet\\\hline\end{array} \xrightarrow{\text{fill with 6}} \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&6\\\hline\end{array}$$

FIGURE 2. Promotion of a standard Young tableau.

To prove Theorem 1 we applied techniques from the theory of lozenge tilings. In particular, we used a variant of *Kuo condensation*, a powerful recurrence method for counting dimer configurations. The specific variant of Kuo condensation we used, which works for regions with a *free boundary*, was recently developed by Ciucu in [6].

There are two features of Theorem 1 which are even more interesting than the result itself.

The first is that there are indications the theorem has a connection to algebra. It is well known that the plane partitions in $\mathcal{PP}^m([a]\times[b])$ index an irreducible representation of the general linear Lie algebra $\mathfrak{gl}(a+b)$, and MacMahon's product formula counting these plane partitions can thus be deduced from the *Weyl dimension formula*. Similarly, the order polynomial product formulas for the other previously studied shapes and shifted shapes come from representations of the Lie algebras $\mathfrak{so}(2n+1)$ and $\mathfrak{sp}(2n)$. Soichi Okada has discovered a remarkable extension of Theorem 1 involving characters of classical Lie algebras and their variants (see [20]). Thus, there is compelling evidence that some deeper representation theory should also "explain" Theorem 1.

The second interesting feature is the method by which Theorem 1 was discovered: namely, via the heuristic (*) connecting product formulas to poset dynamics. As I'll explain below, the shifted double staircase was already known to have good dynamics.

1.2. **Poset dynamics: periodicity.** Let me now introduce some dynamics on poset objects.

Recall that linear extensions of posets of Young diagram shape can be identified with *Standard Young Tableaux (SYTs)*. *Promotion*, denoted Pro, is an invertible operator on SYTs which can briefly be described as follows: first we delete the square containing 1; then we "slide" the remaining squares into the hole thus created in such a way as to preserve standardness; then we decrement the value of all entries; and finally we fill the remaining hole with the maximum value n. This is depicted in Figure 2. Promotion was first defined and studied, in conjunction with a related involutive operator on tableaux called *evacuation*, by Schützenberger. The initial motivation was the close connection of these two operators to the *Robinson–Schensted–Knuth (RSK) correspondence*. The "sliding" procedure which goes into the definition of promotion is called *jeu de taquin*, and it was used by Schützenberger to establish many fundamental results in tableaux theory. There is a straightforward generalization, also due to Schützenberger, of promotion and evacuation to the linear extensions of any poset [38].

For various posets, promotion is known to have algebraic models in terms of reduced words in Coxeter groups, Kazhdan–Lusztig theory, webs, crystals, the Wronski map on the Grassmannian, etc. These models are part of what make promotion so interesting to study.

We want to study promotion from a dynamical perspective. *Periodicity* is one of the most fundamental properties of a dynamical system. Invertible maps on finite sets are trivially periodic, so when we investigate "periodicity" for the invertible operators in dynamical algebraic combinatorics, often what we are really interested in is showing that they have small and predictable orders.

It turns out that the "right" power of promotion to look at for good behavior is the $(\#P)$th power. Schützenberger's foundational work on jeu de taquin implies that for $P=[a]\times[b]$ a rectangle, $\mathrm{Pro}^{\#P}$ is the identity. In their investigation of reduced words in the symmetric group, Edelman and Greene [8] showed that for P a staircase, $\mathrm{Pro}^{\#P}$ is the transposition symmetry. Finally, Haiman [13]

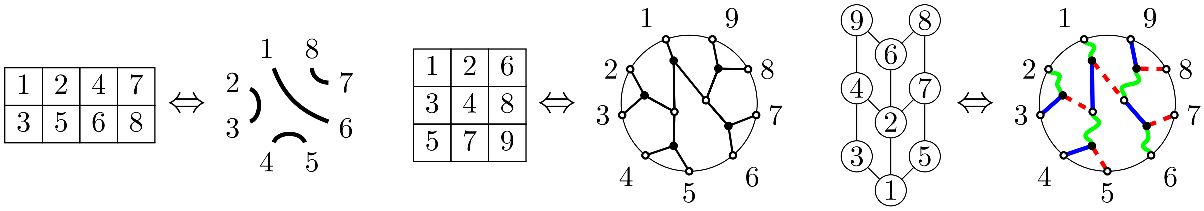
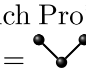
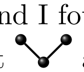


FIGURE 3. Diagrammatic representations of linear extensions of $[2] \times [n]$, $[3] \times [n]$, $V(n)$.

showed that for P a shifted trapezoid or shifted double staircase, $\mathrm{Pro}^{\#P}$ is the identity. Moreover, in a follow-up paper with Kim [14], Haiman showed that these four families are the *only* shapes and shifted shapes for which $\#P$ is the identity or a poset automorphism. Haiman's result about the shifted double staircase, together with the heuristic (*), inspired me to conjecture Theorem 1.

Let me now explain how I have also successfully used (*) in the "opposite" direction, i.e., to discover a family of posets with good dynamical behavior.

In a 2009 survey on promotion and evacuation, Stanley [41] asked whether there were *any* other families of posets, beyond those classified by Haiman [13, 14] in the early 90s, for which $\mathrm{Pro}^{\#P}$ could be understood. Very recently, Martin Rubey and I found such a family. Let $V(n) :=$  $\times [n]$ be the product of the 3-element "V"-shaped poset  and the n-element chain $[n]$. We showed:

Theorem 2 (Hopkins–Rubey [23])**.** *For $P = V(n)$, $\mathrm{Pro}^{\#P}$ is the horizontal reflection symmetry.*

The linear extensions of $V(n)$ were first studied by Kreweras [26]. He gave a remarkable product formula for their enumeration. Later, Kreweras and Niederhausen [25] showed that in fact the entire order polynomial of $V(n)$ has a product formula. This result of Kreweras–Niederhausen, combined with the heuristic (*), is how I was led to Theorem 2.

The linear extensions of $V(n)$ correspond to certain walks in $\mathbb{Z}^2$ called *Kreweras walks*. In the 2000s, Kreweras walks became a fundamental example of "walks with small steps in the quarter plane," a category of lattice walks which was systematically analyzed by Bousquet-Mélou and Mishna, and others, using the *kernel method* from analytic combinatorics. Hence the poset $V(n)$, or at least its linear extensions, have received significant attention in analytic combinatorics.

My work with Rubey suggests that $V(n)$ is also of interest to algebraic combinatorics, because the way we proved Theorem 2 is by encoding linear extensions of $V(n)$ as (edge-colored) webs. *Webs* are certain planar graphs that Kuperberg [27] introduced to study the invariant theory of Lie algebras and their relatives like quantum groups. Prior work of Khovanov–Kuperberg [24], Petersen–Pylyavskyy–Rhoades [31], and Tymoczko [43] showed that promotion of linear extensions of $[3] \times [n]$ corresponds to rotation of $\mathfrak{sl}(3)$-webs. This is an extension of Dennis White's observation [37, §8] that promotion of linear extensions of $[2] \times [n]$ corresponds to rotation of *noncrossing matchings*. We similarly showed that promotion of linear extensions of $V(n)$ corresponds to rotation of their webs. See Figure 3 for these diagrammatic representations.

1.3. **Poset dynamics: orbit structure.** There is another operator on poset objects, called "rowmotion," which also factors into the heuristic (*). *Rowmotion* [4, 30, 42] was originally defined as the operator on the set $\mathcal{J}(P)$ of *order ideals* (downwards closed subsets) of a poset P which sends I to the order ideal generated by $\min(P \setminus I)$. For example, the orbits of rowmotion acting on $\mathcal{J}([2] \times [2])$ are depicted in Figure 4. But there is a natural identification $\mathcal{J}(P) \simeq \mathcal{PP}^1(P)$, and in 2013 Einstein and Propp [9] defined a *piecewise-linear* extension of rowmotion $\mathrm{Row}\colon \mathcal{PP}^m(P) \to \mathcal{PP}^m(P)$ for any $m \in \mathbb{N}$. It is this piecewise-linear rowmotion of P-partitions that I'm interested in.

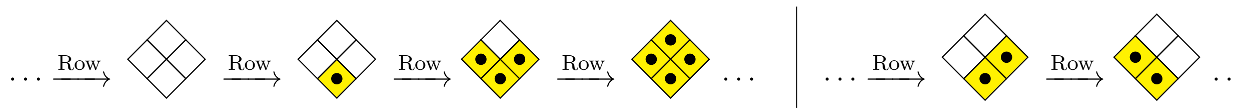


FIGURE 4. The two rowmotion orbits of $\mathcal{J}([2]\times[2])$.

For various posets P, rowmotion of P-partitions is known to have algebraic models in terms of noncrossing partitions in Coxeter groups, maximal parabolic subgroups of minuscule type, canonical bases from quantum groups and cluster algebras, reflection functors for quiver representations, etc. These models are part of what make rowmotion so interesting to study.

We again want to study rowmotion of P-partitions from a dynamical perspective. It turns out that rowmotion only ever exhibits good behavior when P is *graded* (all its maximal chains have the same length $r(P)$), and in this case $\mathrm{Row}^{r(P)+2}$ is the "right" power to look at for good behavior. For several graded posets P it is known that $\mathrm{Row}^{r(P)+2}$ is the identity or an involutive poset automorphism [12]. But even if we know its order, we can ask for finer information about the dynamical behavior of Row: e.g., we can try to describe its entire *orbit structure*.

A very compact way to describe the orbit structure of an invertible map acting on a combinatorial set is via the *cyclic sieving phenomenon (CSP)* [35]. For X a combinatorial set, $C = \langle c\rangle \simeq \mathbb{Z}/n\mathbb{Z}$ a cyclic group of order n acting on X, and $f(q) \in \mathbb{N}[q]$ a polynomial with nonnegative integer coefficients, we say that $(X, C, f(q))$ *exhibits CSP* if $\#X^{c^k} = f(\zeta^k)$ for all $k \in \mathbb{N}$, where $\zeta := e^{2\pi i/n}$ is a primitive nth root of unity.

CSPs involving polynomials $f(q)$ which have a product formula are especially valuable, because they imply that *every* symmetry class of the cyclic action also has a product formula. A beautiful such CSP was established by Rhoades [37]:

Theorem 3 (Rhoades [37])**.** *For $P = [a]\times[b]$ a rectangle,* $\mathrm{Row}\colon \mathcal{PP}^m(P) \to \mathcal{PP}^m(P)$ *has order $r(P)+2 = a+b$, and $(\mathcal{PP}^m(P), \langle\mathrm{Row}\rangle, f(q))$ exhibits CSP, where $f(q)$ is MacMahon's size generating function* (1) *of these plane partitions.*

To prove Theorem 3, Rhoades used the dual canonical basis of $\mathfrak{gl}(a+b)$ representations. A new proof, using a cluster algebra canonical basis, was recently given by Shen and Weng [39].

As part of the heuristic (*) connecting order polynomial product formulas and poset dynamics, I have conjectured a significant extension of Theorem 3:

Conjecture 4 (Hopkins [19])**.** *For the graded posets P for which* $\mathrm{Row}^{r(P)+2}\colon \mathcal{PP}^m(P) \to \mathcal{PP}^m(P)$ *is the identity or an involutive automorphism, $(\mathcal{PP}^m(P), \langle\mathrm{Row}\rangle, \Omega_P(m;q))$ exhibits CSP, where*

$$\Omega_P(m;q) := \prod_{\alpha} \frac{(1-q^{\kappa(m-\alpha)})}{(1-q^{-\kappa\alpha})}$$

is the natural q-analog of the product formula for the order polynomial $\Omega_P(m)$. Here α runs through the roots of $\Omega_P(m)$, and κ is 1 *if these roots are all integral and* 2 *if they are half-integral.*

While I have been unable to resolve Conjecture 4, I was able to prove something morally very similar. The main remaining cases of Conjecture 4 concern certain *"triangular"* posets. Grinberg and Roby [12] showed that rowmotion behaves well for these "triangular" posets by embedding them as subsets of plane partitions in $\mathcal{PP}^m([n]\times[n])$. In this way, Conjecture 4 reduces to the assertion that for various subgroups H of $\langle\mathrm{Row}, \mathrm{Tr}\rangle$ acting on $\mathcal{PP}^m([n]\times[n])$, the numbers of plane partitions fixed by H is given by a CSP-type evaluation. I was able to show, building on Rhoades's Theorem 3, that for any *element* g of $\langle\mathrm{Row}, \mathrm{Tr}\rangle$ acting on $\mathcal{PP}^m([n]\times[n])$, the number of fixed points of g is given by a CSP-type evaluation. Specifically, I showed:

Theorem 5 (Hopkins [17])**.** *We have* $\#\{\pi \in \mathcal{PP}^m([n]\times[n]) \colon \mathrm{Row}^k(\pi) = \mathrm{Tr}(\pi)\} = f(\zeta^k)$ *for all* $k \in \mathbb{N}$*, where* $\zeta := e^{\pi i/n}$ *is a primitive* $(2n)$*th root of unity, and*

$$f(q) = \sum_{\substack{\pi \in \mathcal{PP}^m([n]\times[n]),\\ \mathrm{Tr}(\pi)=\pi}} q^{|\pi|} = \prod_{1\leq i<j\leq n} \frac{(1-q^{2(i+j+m-1)})}{(1-q^{2(i+j-1)})} \cdot \prod_{i=1}^{n} \frac{(1-q^{2i+m-1})}{(1-q^{2i-1})},$$

is MacMahon's size generating function of symmetric plane partitions [2, 28].

To prove Theorem 3, I studied the way that certain involutive automorphisms of the quantized enveloping algebra of $\mathfrak{gl}(a+b)$ behave on the dual canonical basis.

1.4. **Poset dynamics: homomesy.** When studying a dynamical system, it is very desirable to find *invariant functions* (for instance, these can be used to separate orbits). Unfortunately, for the operators from dynamical algebraic combinatorics, it is quite hard in practice to find invariant functions. There has been more success of late in studying a dual notion called "homomesy" [34]. For Φ an invertible operator acting on a combinatorial set X, we say the function $s \colon X \to \mathbb{R}$ is *homomesic* (or *c-mesic*) if the average of s along every Φ-orbit is equal to the same constant $c \in \mathbb{R}$. Homomesy is a "dual" notion to invariance because every function on X can be written uniquely as the sum of an invariant function and a 0-mesic function.

The first example of homomesy is due to Armstrong–Stump–Thomas [3]: they showed that for rowmotion acting on the order ideals of a *root poset*, the *antichain cardinality statistic* $I \mapsto \#\max(I)$ is homomesic. Subsequently, Propp and Roby [34] showed that the antichain cardinality statistic is also homomesic for Row acting on $\mathcal{J}([a]\times[b])$. For example, we see that the average of $\#\max(I)$ along each of the orbits in Figure 4 is $1 = \frac{1}{4}(0+1+2+1) = \frac{1}{2}(1+1)$.

I recently further extended these homomesy results to the piecewise-linear level:

Theorem 6 (Hopkins [18])**.** *For the posets* P *with good behavior of* $\mathrm{Row} \colon \mathcal{PP}^m(P) \to \mathcal{PP}^m(P)$ *(including the rectangle poset, the "triangular" posets, etc.), the piecewise-linear analog of the antichain cardinality statistic is homomesic for this action.*

In fact, I showed that Theorem 6 holds at the *birational* level (see Einstein–Propp [9] for birational poset dynamics). Theorem 6 fits into the philosophy of the heuristic (*) because it shows there is a *universal* homomesy exhibited by all these posets.

In the past, homomesy results have typically been proven by using exact formulas or alternative models for the action. I proved Theorem 6 in a different way: by showing that it follows from a certain structural property of the posets. Specifically, Theorem 6 is the culmination of work I did in a series of papers [5, 15, 16, 18] (the first in collaboration with others) relating the *coincidental down-degree expectations (CDE)* property of posets [36] to rowmotion and homomesy.

2. Past research

For my PhD thesis, I (in collaboration with others) investigated *labeled chip-firing* and *root system chip-firing* [21, 11, 10, 22]. The technical details of this chip-firing work are rather different from my current research. But the two projects are broadly similar in spirit in that they both involved studying a discrete dynamical system using some combinatorics arising from algebra.

3. Future research

My investigation of dynamical algebraic combinatorics, and specifically the heuristic (*), is very much an ongoing project. The main problems I would like to focus on in the near future are:

(1) **Resolve more of the dynamical conjectures arising from the heuristic** (*)**.** These conjectures include Conjecture 4 and related CSP conjectures for promotion, as well as showing that $V(n)$ has good behavior of rowmotion of its P-partitions, showing that certain pairs of *doppelgänger* posets have the same behavior of P-partition rowmotion, and more.

(2) **Investigate possible algebraic explanations for the remarkable combinatorial behavior of these posets.** Some algebraic questions stemming from my research include: what is the representation-theoretic meaning of Okada's character conjecture extending Theorem 1?; what is the invariant-theoretic meaning of the linear extensions of $V(n)$ and their webs?; can we use the dual canonical bases or cluster algebra bases in other types to address more cases of Conjecture 4?

(3) **Find more examples of posets satisfying either side of the heuristic** (*)**.** It seems entirely possible that there are other posets with order polynomial product formulas which have also been "overlooked." Perhaps an exhaustive computer search could even turn up some interesting examples.

(4) **Find formal implications between the properties in the heuristic** (*)**.** There are many spiritual connections between the various objects which enter into the heuristic: for instance, the order polynomial $\Omega_P(m)$ is the Ehrhart polynomial of the order polytope of P, and rowmotion is a piecewise-linear action on this polytope; the linear extensions of P are somehow the $m \to \infty$ "limit" of the P-partitions $\mathcal{PP}^m(P)$; promotion and rowmotion can be defined in very similar ways as compositions of involutions; etc. However, I know of no *direct, formal* implications between any of the properties in this heuristic. It would be very interesting to establish such implications.

Many parts of these problems would be suitable for student research as well. Indeed, I mentored a group of students in a 2019 Research Experience for Undergraduates (REU) at the University of Minnesota, and the problem I gave them dealt with P-partition rowmotion. They were able to resolve a conjecture of mine (specifically, the $m = 1$ case of the aforementioned doppelgänger conjecture) and consequently they wrote a paper [7] which they have submitted for publication. Moreover, previous participants of the Minnesota REU have engaged in related investigations of the cyclic sieving phenomenon (see, e.g., [1]).

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