Root system chip-firing

University of Minnesota Combinatorics Seminar

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MIT

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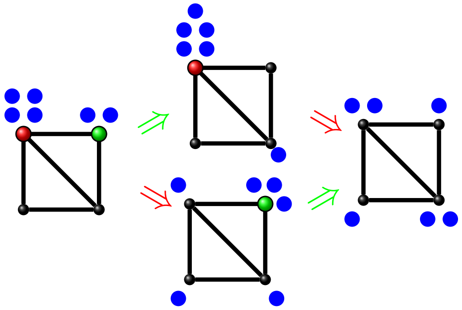
Joint work with Pavel Galashin, Thomas McConville,
and Alexander Postnikov
(and, earlier, with James Propp)

Section 1

Motivation: labeled chip-firing

Classical chip-firing

Classical chip-firing (as introduced by Björner-Lovász-Shor, 1991) is a discrete dynamical system that takes place on a graph. The states are configurations of chips on the vertices. We may *fire* a vertex that has at least as many chips as neighbors, sending one chip to each neighbor:



A key property of this system is that it is *confluent*: from a given initial configuration, either all sequences of firings go on forever, or they all terminate at the same *stable* configuration (called the *stabilization*).

Chip-firing on an infinite path

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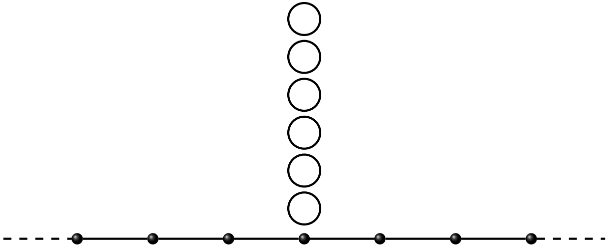

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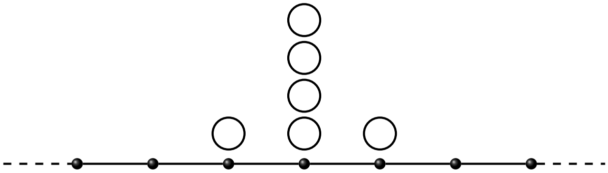

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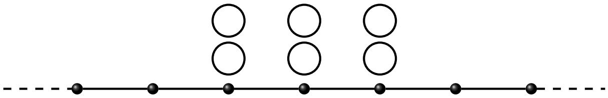

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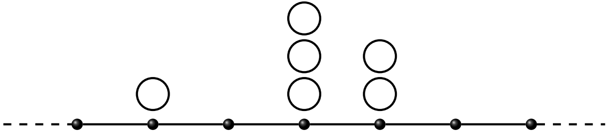

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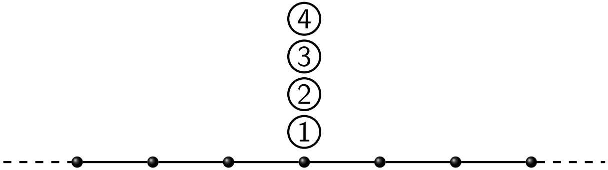
Labeled chip-firing

Jim Propp recently introduced a "labeled" variant of chip-firing on $\mathbb{Z}$. All the chips are now *distinguishable*, given labels from $\mathbb{N}$. Whenever two chips occupy the same vertex we can fire them together, moving the lesser-labeled chip leftwards one vertex and the greater-labeled chip rightwards one vertex:



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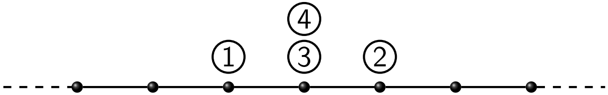

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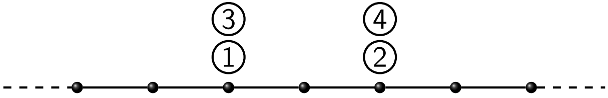

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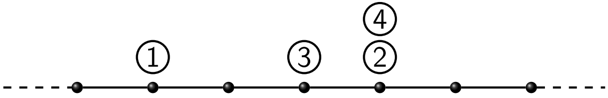

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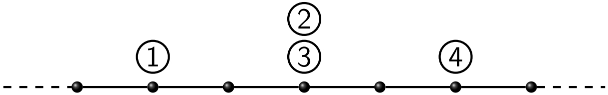

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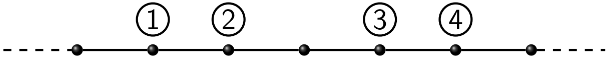

Labeled chip-firing is not confluent in general

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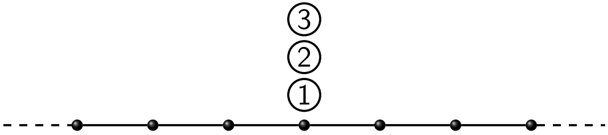

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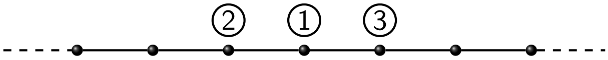

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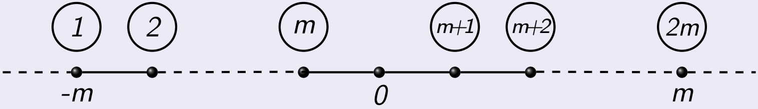
Sorting an even number of chips

Theorem (Hopkins-McConville-Propp, 2017)

Suppose $n = 2m$ is even. Then starting from n labeled chips at the origin, the chip-firing process "sorts" the chips to a unique stable configuration:



A "Type B" version of labeled chip-firing

Consider a modified version of labeled chip-firing on $\mathbb{Z}$ where we allow the following three kinds of moves:

(I) for $i < j$, if (i) and (j) occupy the same vertex, move (i) leftwards one vertex and (j) rightwards one vertex (this is the same as before);

(II) for any i, j, if (i) is at vertex a and (j) is at vertex $-a$, move both (i) and (j) rightwards one vertex;

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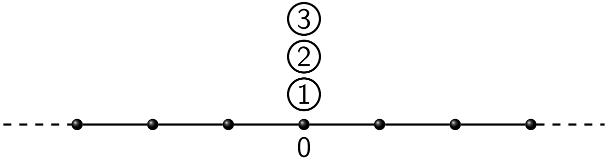

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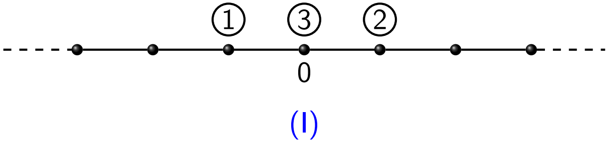

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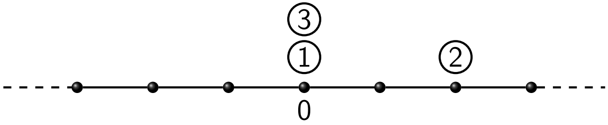

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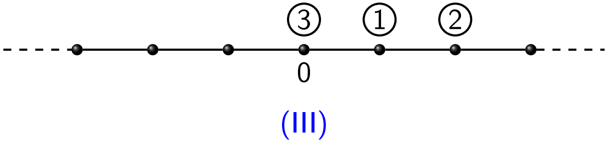

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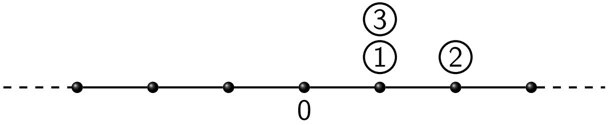

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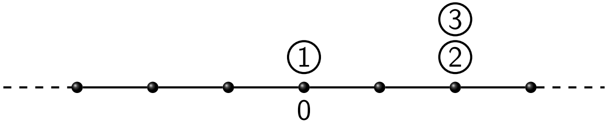

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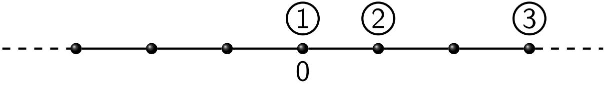

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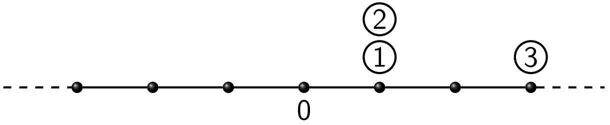

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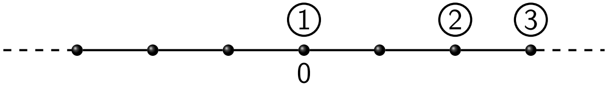

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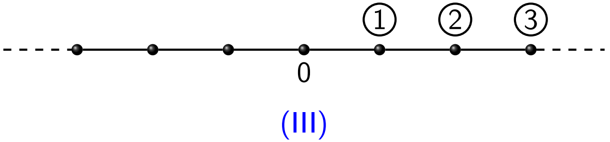
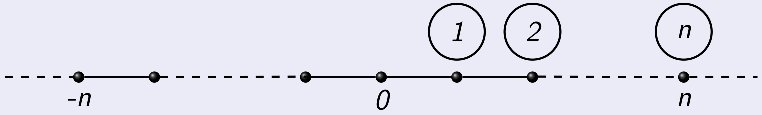


"Type B" sorting

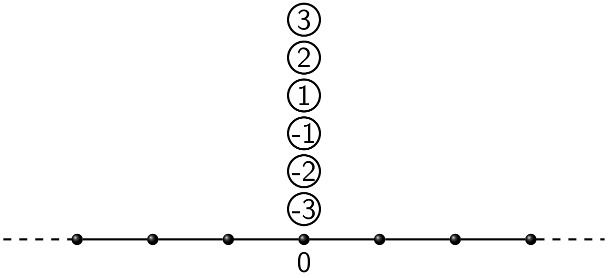
Theorem (Hopkins-McConville-Propp, 2017)

For any n, starting from n labeled chips at the origin, the "Type B" labeled chip-firing process (with moves (I), (II), and (III)) "sorts" the chips to the following unique stable configuration:



Proof of "Type B" sorting via symmetry

Use positive chips (1), (2), ..., (n) and negative chips ($-n$), (-2), ..., (-1). Start with all of them at the origin, and carry out "symmetrical" labeled chip-firing moves (whenever we fire (i) and (j) we also fire ($-i$) and ($-j$)). The way the positive chips evolve corresponds exactly to the "Type B" moves (I), (II), and (III) above:



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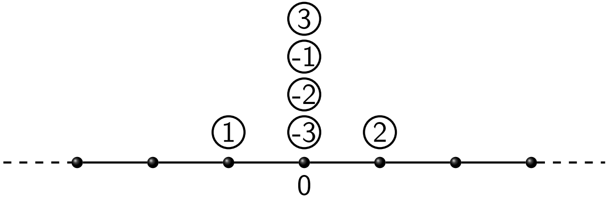

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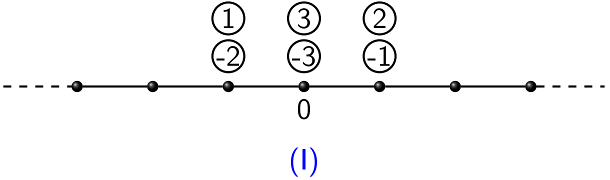

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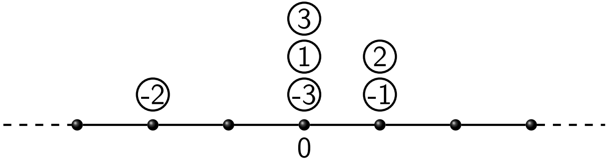

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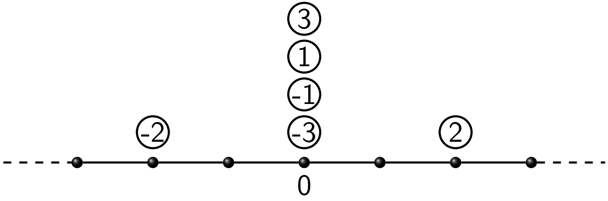

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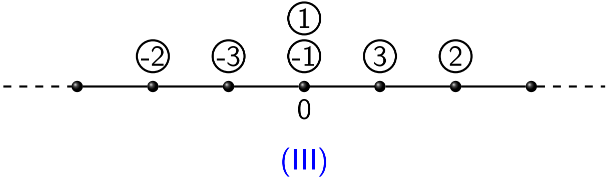


Section 2

Central-firing

Root system reformulation of Propp's labeled chip-firing

For any configuration of n labeled chips, if we set $c := (c_1, \ldots, c_n) \in \mathbb{Z}^n$ where

$$c_i := \text{the position of the chip } \textcircled{i},$$

then, for $1 \leq i < j \leq n$, we are allowed to fire chips $\textcircled{i}$ and $\textcircled{j}$ in this configuration as long as c is orthogonal to $e_j - e_i$; and doing so replaces the vector c by $c + (e_j - e_i)$.

Observation: the vectors $e_j - e_i$ for $1 \leq i < j \leq n$ are exactly the positive roots Φ^+ of the root system Φ of Type A_{n-1}.

Root systems: basic definitions

Let V be a Euclidean vector space with standard inner product $\langle \cdot, \cdot \rangle$. For any $v \in V$, define the *co-vector* $v^\vee := \frac{2}{\langle v, v \rangle} v$, and the *(orthogonal) refletion* $s_v \colon V \to V$ by $s_v(w) := w - \langle w, v^\vee \rangle v$.

Definition

A *(crystallographic) root system* in V is a finite subset $\Phi \subseteq V$ such that:

1. Φ spans V;
2. $s_\alpha(\Phi) = \Phi$ for all $\alpha \in \Phi$;
3. $\mathbb{R}\alpha \cap \Phi = \{\pm\alpha\}$ for all $\alpha \in \Phi$;
4. $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in \Phi$.

The elements of Φ are called *roots*. The *rank* of Φ is $r = \dim(V)$.

The *Weyl group*, denoted W, of Φ is the group generated by the reflections s_α for $\alpha \in \Phi$. By definition, it is a finite reflection group.

Positive roots and lattices

We choose a generic linear functional to separate Φ into *positive* Φ^+ and *negative* Φ^- roots. This also defines a basis $\alpha_1, \alpha_2, \ldots, \alpha_r$ of *simple roots* with the property that every positive root is a nonnegative integral combination of simple roots. The *length* $\ell(w)$ of $w \in W$ is the minimal length of an expression of w as a product of *simple reflections* $s_i := s_{\alpha_i}$.

Two important lattices attached to Φ are the *root lattice* $Q := \mathbb{Z}\Phi$, and the *weight lattice* $P := \{\lambda \in V \colon \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$. By the assumption of crystallography, we have $Q \subseteq P$.

The *fundamental weights* $\omega_1, \ldots, \omega_r$ are defined by $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ and these generate P. A weight is *dominant* if it is a nonnegative sum of fundamental weights. An important dominant weight is the so-called *Weyl vector* $\rho := \sum_{i=1}^r \omega_i = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$.

Minuscule weights are certain distinguished fundamental weights. The minuscule weights together with zero give coset representatives for P/Q.

Classification of root systems

Attached to each root system is a (decorated) graph called the *Dynkin diagram* that records inner products between the simple roots. These lead to a classification of irreducible root systems by the *(Cartan-Killing) types*, which include the *classical types*

$$A_{n-1} := \{\pm(e_i - e_j) \colon 1 \leq i < j \leq n\},$$

$$D_n := A_{n-1} \cup \{\pm(e_i + e_j) \colon 1 \leq i < j \leq n\},$$

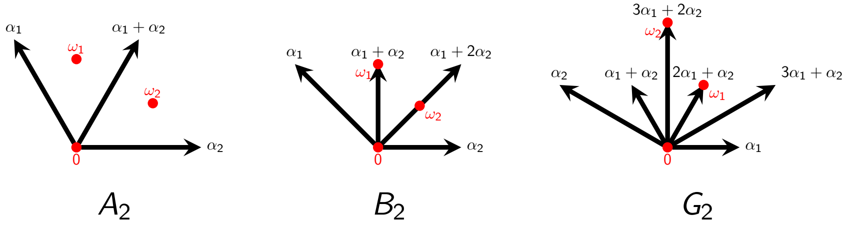
$$B_n := D_n \cup \{\pm e_i \colon 1 \leq i \leq n\},$$

$$C_n := D_n \cup \{\pm 2e_i \colon 1 \leq i \leq n\},$$

as well as the *exceptional types* G_2, F_4, E_6, E_7, E_8.

Rank 2 root systems

The following are the positive roots and fundamental weights of the irreducible rank 2 root systems:



Central-firing for root systems

The description of labeled chip-firing in terms of positive roots of A_{n-1} generalizes naturally to any root system Φ: for a weight $\lambda \in P$, we allow the firing moves $\lambda \to \lambda + \alpha$ for a positive root $\alpha \in \Phi^+$ whenever λ is orthogonal to α.

We call the resulting system the *central-firing* process for Φ (because we allow firing from a weight λ when λ belongs to the Coxeter hyperplane arrangement, which is a central arrangement).

You can check that the previously described "Type B" labeled chip-firing really is central-firing for $\Phi = B_n$. Other classical types have similar description of central-firing in terms of chips.

Confluence of central-firing

Question

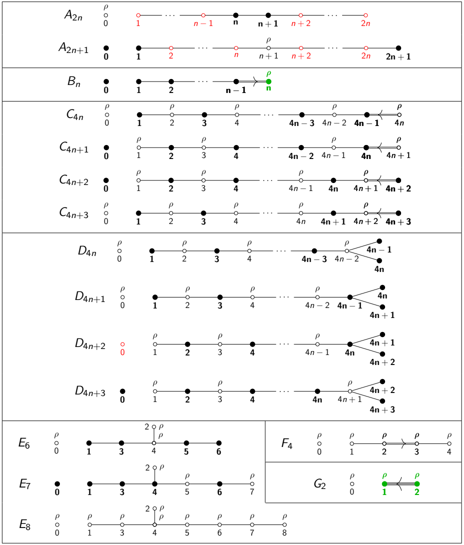
For any root system Φ and weight $\lambda \in P$, when is central-firing confluent from λ?

Answer: it's complicated.

Classification of confluence for origin/fundamental weights

Conjecture

Confluence of central-firing from λ for $\lambda = 0$ or λ a fundamental weight is determined by the table on the right. To first order, central-firing is confluent from λ iff $\lambda \neq \rho$ modulo Q. Exceptions to this are in red or green.



This conjecture is proved in some but not all cases (e.g. for $\lambda = 0$ and $\Phi = A_n$ or B_n, it follows from H.-M.-P. theorems above).

Confluence of central-firing modulo the Weyl group

Theorem

For any root system Φ, and from any initial weight λ, central-firing is confluent modulo the action of the Weyl group W.

In Type A the Weyl group is the symmetric group, so modding out by the Weyl group is the same as forgetting the labels of chips. Thus this theorem gives a generalization of *unlabeled* chip-firing on a line to any root system.

Note: this is very different from the Cartan matrix chip-firing studied by Benkart-Klivans-Reiner, 2016 (e.g., for $\Phi = A_{n-1}$, ours corresponds to chip-firing of n chips on the infinite path, whereas B.-K.-R. corresponds to chip-firing of any number of chips on the n-cycle).

Unlabeled central-firing for simply laced root systems

Suppose Φ is *simply laced*, i.e., its Dynkin diagram Γ is just an undirected graph with nodes $1, 2, \ldots, r$. Consider the following process on the set of labelings $\gamma\colon \Gamma \to \mathbb{N}$ of the nodes of Γ by nonnegative integers:

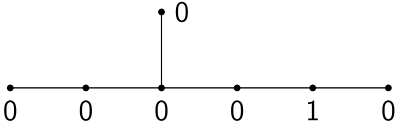
1. choose any connected component X of $\Gamma[\{i\colon \gamma(i) = 0\}]$;
2. extend X to an *affine Dynkin diagram* $\widetilde{X}$ in a unique way;
3. for each edge $(0, i)$, where 0 is the "affine node" of $\widetilde{X}$, add 1 to the label of i;
4. for each $j \in \Gamma \setminus X$ with j adjacent to i for some $i \in X$, decrease the label of j by 1.

Theorem

Central-firing modulo the Weyl group is the same process as the one defined by the above moves, where we represent an orbit $W.\lambda$ for a dominant weight $\lambda = \sum_{i=1}^{r} c_i\omega_i$ by the function $\gamma(i) = c_i$.

Unlabeled central-firing example

Here is an example of a few unlabeled central-firing moves...



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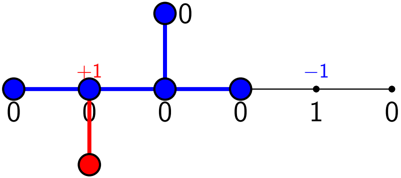

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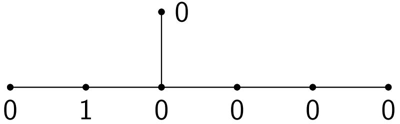

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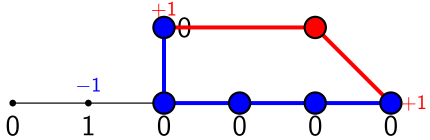

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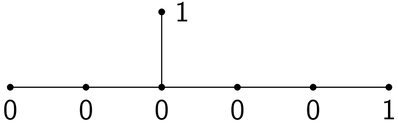
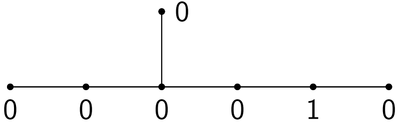


Another unlabeled central-firing example

Let's try that same example with some other choices...



(We see the "abelian property" here.)

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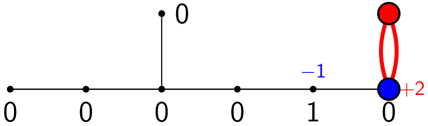

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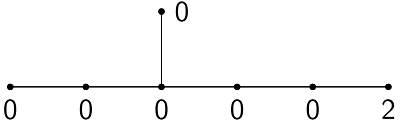

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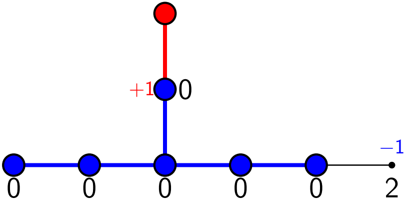

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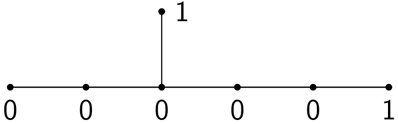
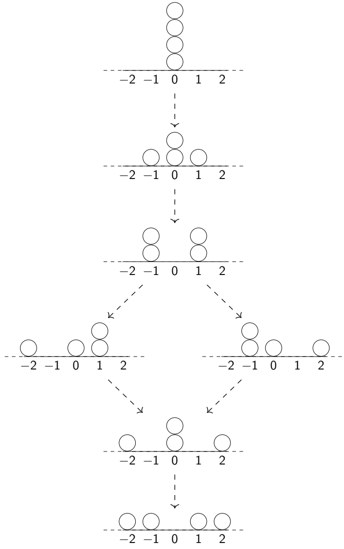
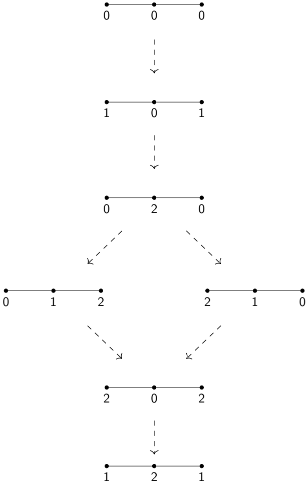


(We see the "abelian property" here.)

Unlabeled central-firing versus chip-firing





Section 3

Interval-firing: confluence

Interval-firing

Central-firing allows the firing move $\lambda \to \lambda + \alpha$ whenever $\langle \lambda, \alpha^\vee \rangle = 0$ for $\lambda \in P$ and $\alpha \in \Phi^+$. We found remarkable "deformations" of this process.

For any $k \in \mathbb{N}$, define the *symmetric interval-firing process* by

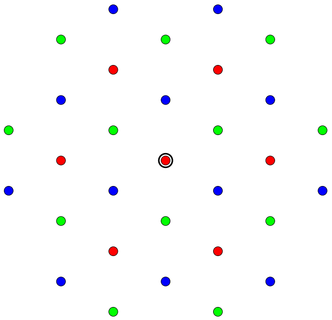
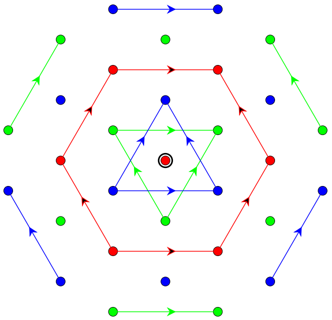
$$\lambda \to \lambda + \alpha \qquad \text{if } \langle \lambda, \alpha^\vee \rangle \in \{-k-1, -k, \ldots, k-1\}$$

and the *truncated interval-firing process* by

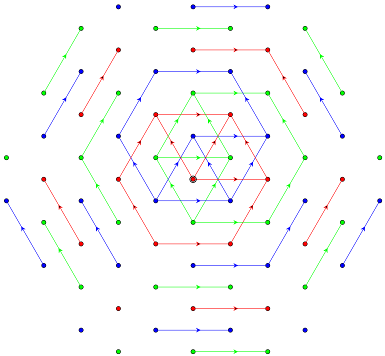
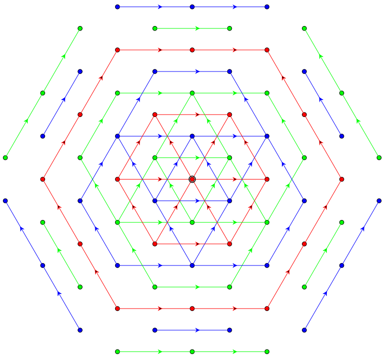
$$\lambda \to \lambda + \alpha \qquad \text{if } \langle \lambda, \alpha^\vee \rangle \in \{-k, -k+1 \ldots, k-1\}.$$

(These are analogous to the (extended) $\Phi^\vee$-Catalan and $\Phi^\vee$-Shi hyperplane arrangements, respectively. The symmetric closure of the symmetric process is W-invariant, explaining its name.)

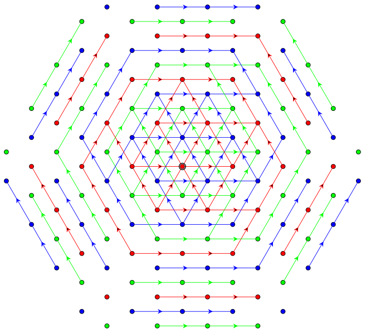
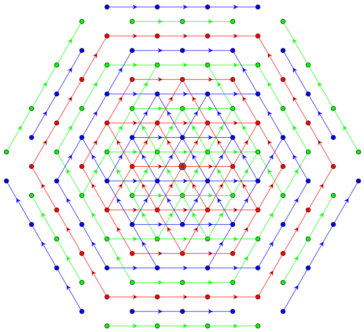
Pictures of interval-firing for A_2

k	truncated	symmetric
0		

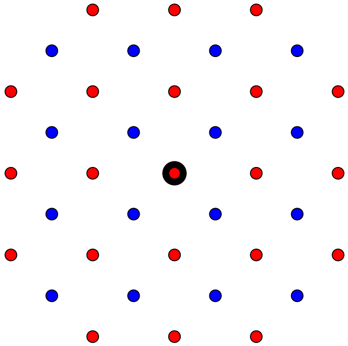
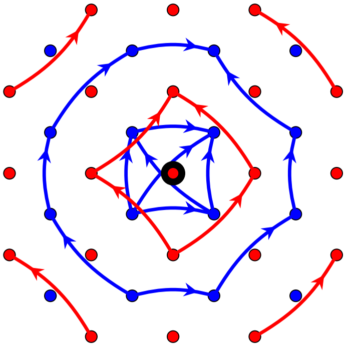
Pictures of interval-firing for A_2

k	truncated	symmetric
1		

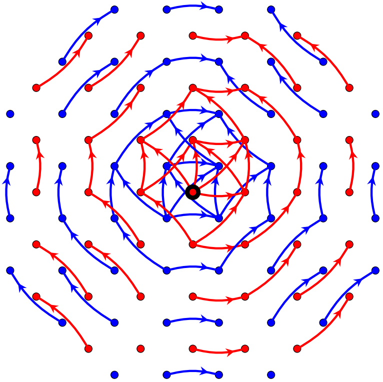
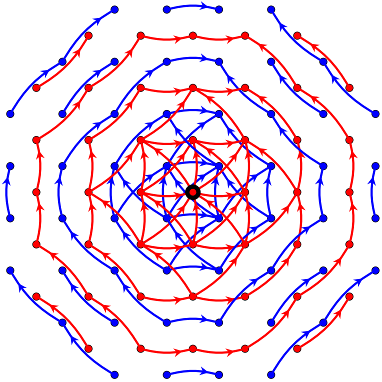
Pictures of interval-firing for A_2

k	truncated	symmetric
2		

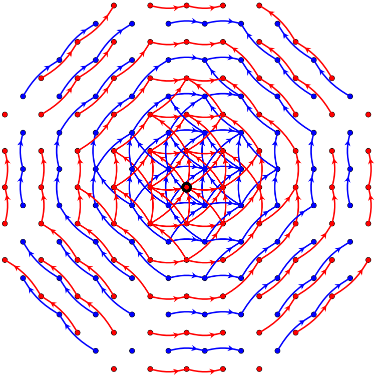
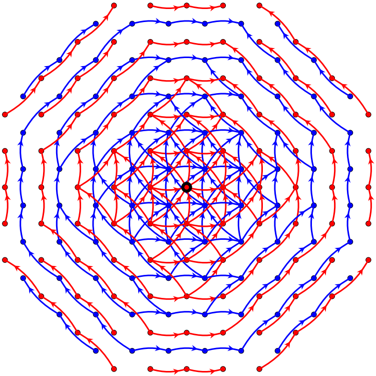
Pictures of interval-firing for B_2

k	truncated	symmetric
0		

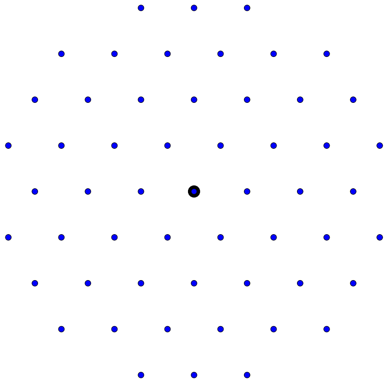
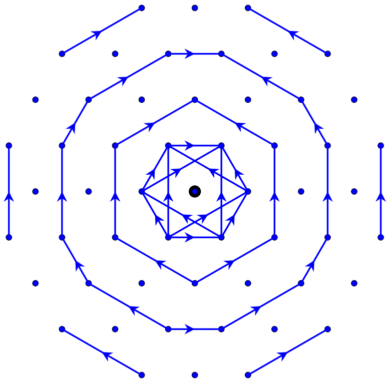
Pictures of interval-firing for B_2

k	truncated	symmetric
1		

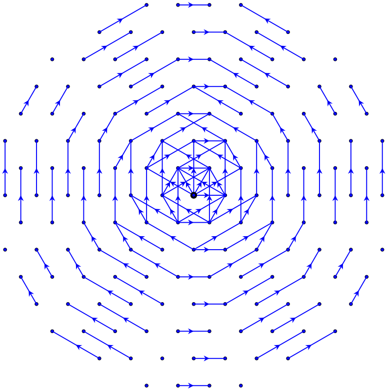
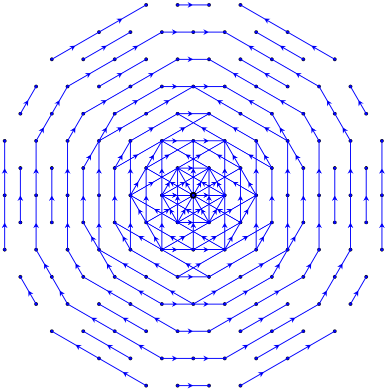
Pictures of interval-firing for B_2

k	truncated	symmetric
2		

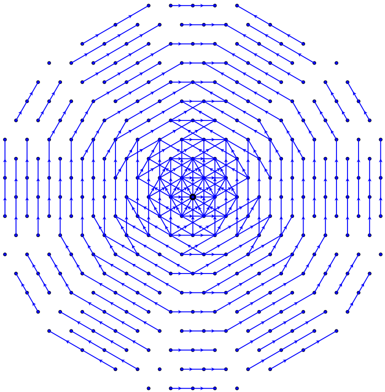
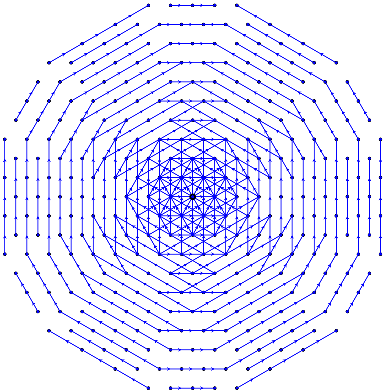
Pictures of interval-firing for G_2

k	truncated	symmetric
0		

Pictures of interval-firing for G_2

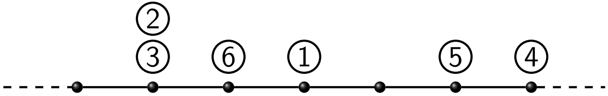
k	truncated	symmetric
1		

Pictures of interval-firing for G_2

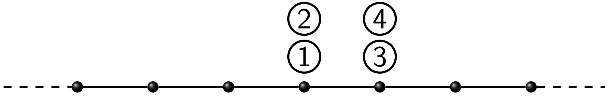
k	truncated	symmetric
2		

Interval-firing in Type A via chips

When $\Phi = A_{n-1}$, we can interpret interval-firing in terms of chips. The smallest nontrivial case is symmetric $k = 0$ interval-firing, which has $\lambda \to \lambda + \alpha$ for $\lambda \in P, \alpha \in \Phi^+$ when $\langle \lambda, \alpha^\vee \rangle = -1$. This corresponds to allowing *(adjacent) transpositions* of ⓘ and ⓙ if they're out of order:



Here confluence is obvious. The next smallest case is truncated $k = 1$ interval-firing, which has $\lambda \to \lambda + \alpha$ when $\langle \lambda, \alpha^\vee \rangle \in \{-1, 0\}$. This corresponds to allowing transpositions as well as the usual labeled chip-firing moves:



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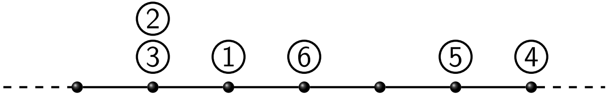

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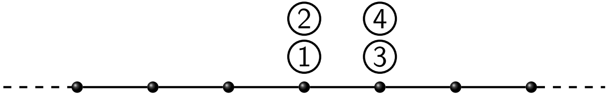

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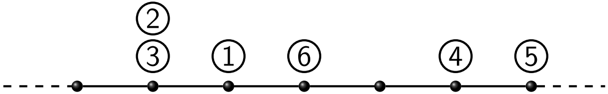

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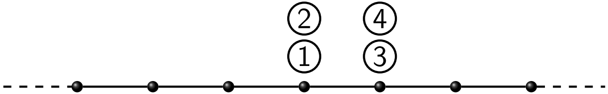

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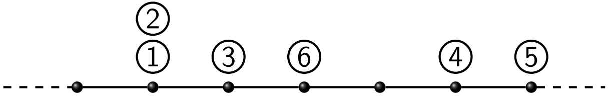

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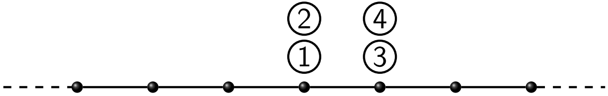

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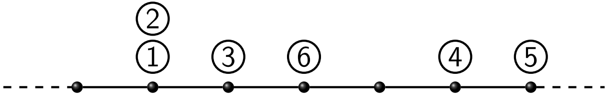

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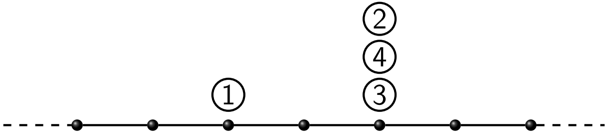

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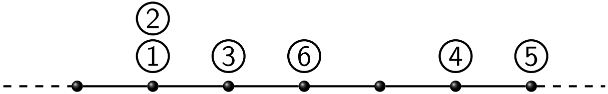

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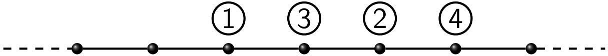

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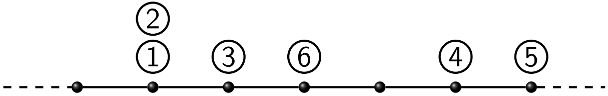

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Interval-firing is confluent

Theorem

For any root system Φ, and any $k \geq 0$, the symmetric and truncated interval-firing processes are confluent (from all initial weights).

I'll now go over some (geometric) ideas that go into the proof. The main ingredient is a formula for *traverse lengths of permutohedra*.

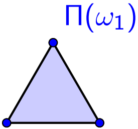
Traverse lengths of permutohedra

For $\lambda \in P$, we define the *(W-)permutohedron* $\Pi(\lambda) := \mathrm{ConvexHull}\, W(\lambda)$. We use $\Pi^Q(\lambda) := \Pi(\lambda) \cap (Q + \lambda)$ to denote the lattice points in $\Pi(\lambda)$.

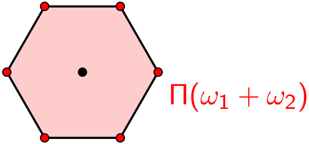
An *α-string* of length k is a collection $\{\mu, \mu - \alpha, \ldots, \mu - k\alpha\} \subseteq P$. An *$\alpha$-traverse in* $\Pi(\lambda)$ is a **maximal** (by containment) α-string inside $\Pi^Q(\lambda)$. We define $\ell_\lambda(\alpha)$, the *traverse length of* $\Pi(\lambda)$ *in direction* α, to be the **minimum length** of an α-traverse in $\Pi(\lambda)$.

Examples for $\Phi = A_2$:

$\ell_{\omega_1}(\alpha) = 0 \; \forall \alpha \in \Phi$



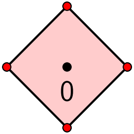
$\ell_{\omega_1+\omega_2}(\alpha) = 1 \; \forall \alpha \in \Phi$



Intuition: the minimal length α-traverse should be an *edge* of $\Pi(\lambda)$.

“Funny weights” and traverse length formula

Counterexample to (almost correct) intuition: for $\Phi = B_2$, $\ell_{\omega_1}(\alpha_1) = 0$



Definition

If Φ is not simply laced, then there are l and s such that the long simple root α_l and short simple root α_s are adjacent in the Dynkin diagram. We say the dominant weight $\lambda = \sum_{i=1}^{r} c_i\omega_i$ is *funny* if $c_s = 0$, $c_l \geq 1$, and $c_i \geq c_l$ for all long α_i. (No weight is funny for simply laced Φ.)

Theorem

For a dominant weight $\lambda = \sum_{i=1}^{r} c_i\omega_i$, set $m_\lambda(\alpha) := \min\{c_i : \alpha_i \in W(\alpha)\}$.

$$\ell_\lambda(\alpha) = \begin{cases} m_\lambda(\alpha) - 1 & \text{if } \lambda \text{ is funny and } \alpha \text{ is long,} \\ m_\lambda(\alpha) & \text{otherwise.} \end{cases}$$

Relevance of traverse length to interval-firing

Lemma

$$\ell_\lambda(\alpha) = \min\{\langle \mu, \alpha^\vee \rangle \colon \mu \in \Pi^Q(\lambda), \mu + \alpha \notin \Pi^Q(\lambda)\}$$

Proof.

Let $\{\mu, \mu - \alpha, \ldots, \mu - k\alpha\}$ be an α-traverse in $\Pi^Q(\lambda)$. By the W-invariance of $\Pi(\lambda)$ we have $s_\alpha(\mu - i\alpha) = \mu - (k-i)\alpha$ for $i = 0, \ldots, k$. Thus in particular

$$\mu - \langle \mu, \alpha^\vee \rangle \alpha = s_\alpha(\mu) = \mu - k\alpha,$$

so $\langle \mu, \alpha^\vee \rangle = k$. By definition $\ell_\lambda(\alpha)$ is the minimal such k. □

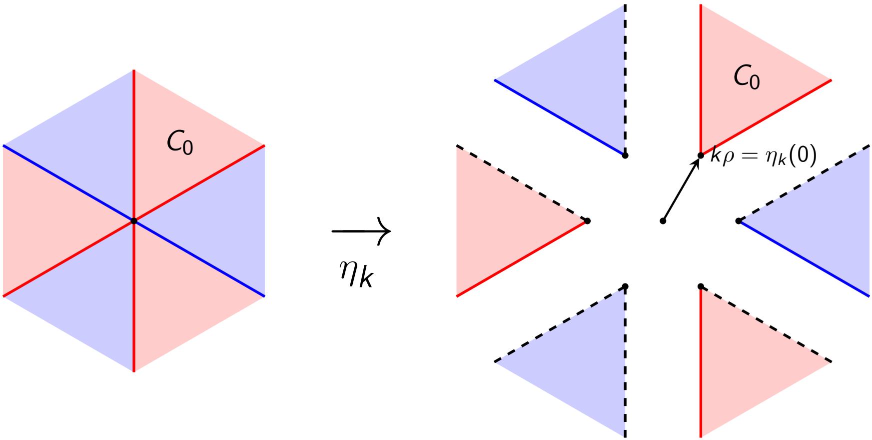
So the formula for traverse lengths says that interval-firing processes get "trapped" inside certain permutohedra, leading to a proof of confluence.

Section 4

Interval-firing: stabilizations

The map η_k

Define $\eta_k \colon P \to P$ by $\eta_k(\lambda) = \lambda + w_\lambda(k\rho)$, where $w_\lambda \in W$ is of minimal length such that $w_\lambda^{-1}(\lambda)$ is dominant.



The stable points of interval-firing

Lemma

The stable points of symmetric interval-firing are

$$\{\eta_k(\lambda) \colon \lambda \in P, \langle \lambda, \alpha^\vee \rangle \neq -1 \textit{ for all } \alpha \in \Phi^+\},$$

and the stable points of truncated interval-firing are

$$\{\eta_k(\lambda) \colon \lambda \in P\}.$$

Stabilization maps and Ehrhart-like polynomials

For $k \geq 0$, define the stabilization maps $s_k^{\mathrm{sym}}, s_k^{\mathrm{tr}} \colon P \to P$ by

$$s_k^{\mathrm{sym}}(\mu) = \lambda \Leftrightarrow \text{the symmetric interval-firing stabilization of } \mu \text{ is } \eta_k(\lambda);$$
$$s_k^{\mathrm{tr}}(\mu) = \lambda \Leftrightarrow \text{the truncated interval-firing stabilization of } \mu \text{ is } \eta_k(\lambda).$$

We want to show that there exists (*Ehrhart-like*) polynomials $L_\lambda^{\mathrm{sym}}(k)$, $L_\lambda^{\mathrm{tr}}(k)$ such that for all $k \geq 0$,

$$\#(s_k^{\mathrm{sym}})^{-1}(\lambda) = L_\lambda^{\mathrm{sym}}(k);$$
$$\#(s_k^{\mathrm{tr}})^{-1}(\lambda) = L_\lambda^{\mathrm{tr}}(k).$$

Theorem

For all Φ and all $\lambda \in P$, the symmetric polynomial $L_\lambda^{\mathrm{sym}}(k)$ exists.

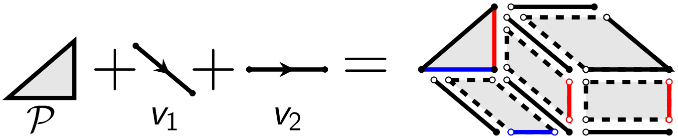
Theorem

For simply laced Φ and all $\lambda \in P$, the truncated polynomial $L_\lambda^{\mathrm{tr}}(k)$ exists.

Lattice points in dilated zonotope plus fixed polytope

Theorem

For any lattice polytope $\mathcal{P}$ and lattice zonotope $\mathcal{Z}$, the number of lattice points in $\mathcal{P} + k\mathcal{Z}$ is given by a polynomial (with $\mathbb{Z}_{\geq 0}$ coefficients) in k.



Corollary

For any dominant $\lambda \in P$, $\#\Pi^{Q}(\lambda + k\rho)$ is given by a polynomial (with $\mathbb{Z}_{\geq 0}$ coefficients) in k.

The previous corollary leads to the existence of the $L^{\mathrm{sym}}_{\lambda}(k)$.

Decomposing connected components of interval firing

For fixed k, the firing moves in truncated interval-firing are a subset of the moves in symmetric interval-firing, so the symmetric "connected components" break into truncated components. Similarly, the $k-1$ symmetric moves are a subset of the k truncated moves, so the truncated components break into $k-1$ symmetric components. Ideally the way that these break up would be consistent with η_k. This is indeed the case.

Lemma

For all Φ, $\mu \in P$, and $k \geq 0$, $s_k^{\mathrm{sym}}(\mu) = s_0^{\mathrm{sym}}(s_k^{\mathrm{tr}}(\mu))$.

Lemma

For simply laced Φ, $\mu \in P$, and $k \geq 1$, $s_k^{\mathrm{tr}}(\mu) = s_1^{\mathrm{tr}}(s_{k-1}^{\mathrm{sym}}(\mu))$.

The previous lemma leads to the existence of the $L_\lambda^{\mathrm{tr}}(k)$. The simply laced assumption is technical and we expect it can be dropped.

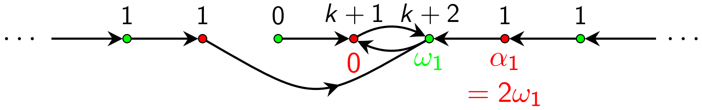
Sizes of fibers of iterates of a function

By iterating the previous two lemmas we obtain (for simply laced Φ) that

$$s_k^{\mathrm{sym}}(\mu) = (s_1^{\mathrm{sym}})^k(\mu).$$

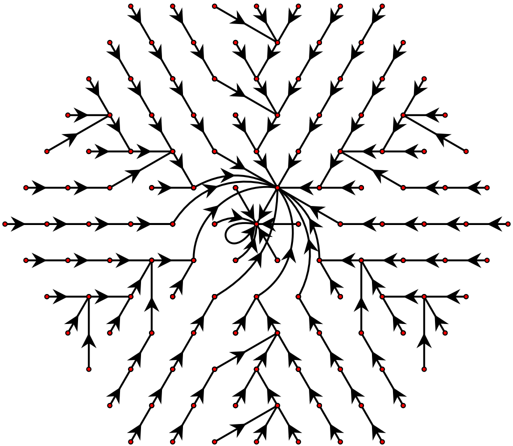
But we know that $\#(s_k^{\mathrm{sym}})^{-1}(\mu) = L_\lambda^{\mathrm{sym}}(k)$ is given by a polynomial. So we conclude that $s_1^{\mathrm{sym}}\colon P \to P$ is a function for which the **sizes of fibers of iterates are all given by polynomials.**

Example for $\Phi = A_1$:



Another iteration example

For $\Phi = A_2$, it turns out that $\rho \in Q$ and hence s_1^{sym} descends to a map $s_1^{\mathrm{sym}} \colon Q \to Q$. Here is that map:



Positivity conjecture

Conjecture

For all Φ and all $\lambda \in P$, the polynomials $L^{\mathrm{sym}}_{\lambda}(k)$ and $L^{\mathrm{tr}}_{\lambda}(k)$ exist and have coefficients in $\mathbb{Z}_{\geq 0}$.

Our proofs show the coefficients are in $\mathbb{Z}$. We can prove positivity of coefficients only when λ is zero or a minuscule weight. A reasonable amount of computational evidence backs up this conjecture.

Future directions

- Prove the Ehrhart-like polynomial positivity conjecture. (Hard?)
- Is there a connection between interval-firing and the *quasi-invariants* of W? (For simplicity I didn't define this but for non-simply laced Φ there is a "two parameter" version of interval-firing.)
- Is there a connection between interval-firing and the extended $\Phi^\vee$-Catalan and $\Phi^\vee$-Shi arrangements (known to be free, affirming conjecture of Edelman-Reiner, by work of Yoshinaga & Terao)?
- Is there a more conceptual proof that central-firing modulo the Weyl group is confluent (our proof uses Newman's lemma in an unilluminating way)?
- Further understand the pattern of central-firing confluence.

Thank you!

References:

- Hopkins, McConville, Propp. "Sorting via chip-firing." Electronic Journal of Combinatorics, 24(3), 2017.
- Galashin, Hopkins, McConville, Postnikov. "Root system chip-firing I: Interval-firing." `arXiv:1708.04850`.
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