Lecture 25

Real Options

Real Options

We can try to use our knowledge of derivatives to help value real objects, such as buildings, land, and equipment.

Often there are options embedded in these investment opportunities, and often difficult to price well.

Capital Investment Appraisal

Traditionally one values a potential capital investment project is known as **net present value** (NPV). The NPV of a project is the present value of its expected future incremental cash flows.

The discount rate used to calculate the present value is a **risk-adjusted** discount rate, chosen to reflect the risk of the project.

As the riskiness of the project increases, the discount rate also increases.

Mean-Reverting Process

This model is too simple, since most commodities follow mean-reverting processes. They then to get pulled back to a central value. A more realistic process than

$$\frac{dS}{S} = \mu(t)dt + \sigma dz$$

would be the process

$$d\ln S = [\theta(t) - a\ln S]\,dt + \sigma dz \tag{1}$$

This incorporates mean reversion and is analogous to the lognormal process assumed for the short-term interest rate (Hull-White). We can then construct a trinomial tree for S and determine the value of $\theta(t)$ such that $F(t) = \hat{E}\,[S(t)]$.

Example: We build a three-step tree for oil. Suppose the spot price for oil is $20 per barrel and the 1-year, 2-year, and 3-year futures prices are $22, $23, and $24, respectively. Suppose that $a = 0.1$ and $\sigma = 0.2$.

We first define a variable X that is initially zero and follows the process

$$dX = -adt + \sigma dz$$

We use the procedure to build a trinomial tree for X.

- The mean change in X in time $-aX\Delta t$ and the variance of the change is $\sigma^2 \Delta t$.
- Branching must satisfy the following three equations for risk-neutral evaluation at (i, j) is
  1. $p_u \Delta X - p_d \Delta X = -aj\Delta X \Delta t$
  2. $p_u \Delta X^2 + p_d \Delta X^2 = \sigma^2 \Delta t + a^2 j^2 \Delta X^2 \Delta t^2$
  3. $p_u + p_m + p_d = 1$

  and choosing $\Delta X = \sigma\sqrt{3\Delta t}$ we find
  1. $p_u = \frac{1}{6} + \frac{1}{2}\left(a^2 j^2 \Delta t^2 - aj\Delta t\right)$
  2. $p_m = \frac{2}{3} - a^2 j^2 \Delta t^2$
  3. $p_d = \frac{1}{6} + \frac{1}{2}\left(a^2 j^2 \Delta t^2 + aj\Delta t\right)$

For us this requires at say (0,0):

$$p_u = \frac{1}{6} + \frac{1}{2}\left[0^2 \times 1^2 - 0 \times 1 \times 1\right] = \frac{1}{6}$$

$$p_m = \frac{2}{3} + 0.1^2 0^2 \times 1 = \frac{2}{3}$$

$$p_d = \frac{1}{6} + \frac{1}{2}\left[0^2 \times 1^2 + 0 \times 1 \times 1\right] = \frac{1}{6}$$

and at say (1,1)

$$p_u = \frac{1}{6} + \frac{1}{2}\left[0.1^2 \times 1^2 \times 1^2 - 0.1 \times 1 \times 1\right] = 0.1217$$

$$p_m = \frac{2}{3} - 0.1^2 \times 1 = 0.6566$$

$$p_d = \frac{1}{6} + \frac{1}{2}\left[0.1^2 \times 1^2 + 0.1 \times 1 \times 1\right] = 0.2217$$

and at say (2,2)

$$p_u = \frac{1}{6} + \frac{1}{2}\left[0.1^2 \times 2^2 \times 1^2 - 0.1 \times 2 \times 1\right] = 0.8867$$

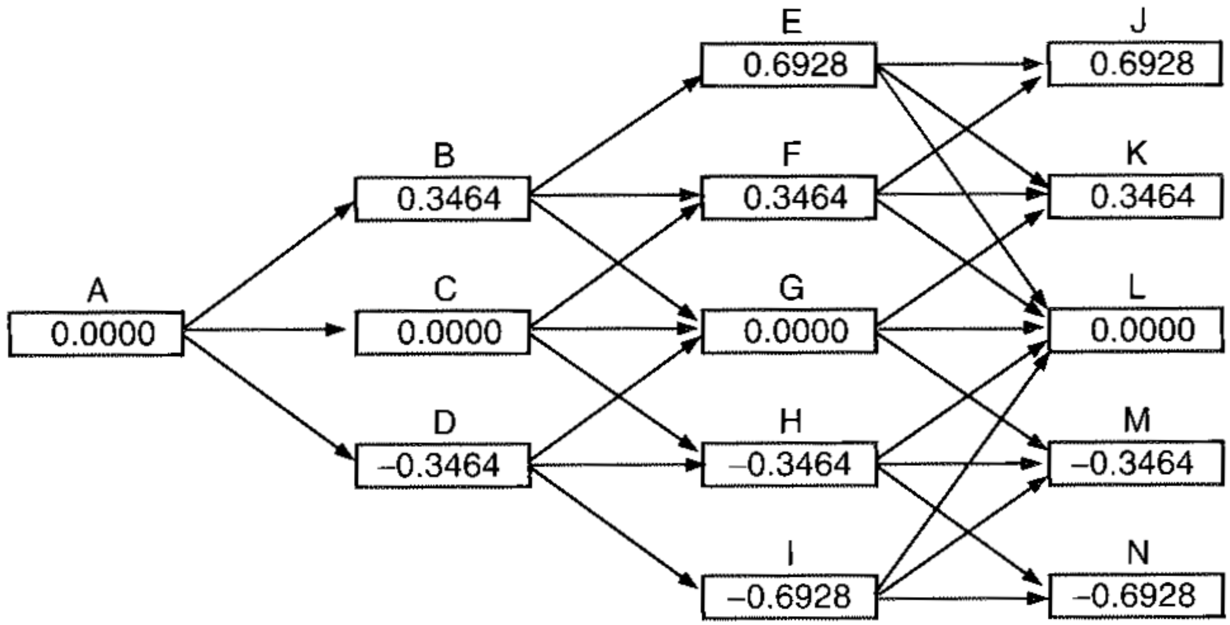
$$p_m = \frac{2}{3} - 0.1^2 2^2 \times 1 = 0.0266$$

$$p_d = \frac{1}{6} + \frac{1}{2}\left[0^2 \times 1^2 + 0 \times 1 \times 1\right] = 0.0867$$

We need to control the size of the probabilities. If they rise above $\frac{0.816}{a\Delta t}$ or go below $\frac{0.184}{a\Delta t}$ then need to modify and choose a different branching at those values (for numerical stability).

Next we compute the displacement: $\Delta X = \sigma\sqrt{3\Delta t} = 0.2\sqrt{3} = 0.3464.$

Figure 31.1 Tree for X. Constructing this tree is the first stage in constructing a tree for the spot price of oil, S. Here p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Next the variable $\ln S$ follows the same process as X except for a time-dependent drift. We can convert the tree for X to a tree for $\ln S$ by displacing the positions of nodes.

The tree can be adjusted accordingly:

- The initial node corresponds to an oil price of 20, so the displacement for that node is $\ln 20$. Supose that the displacements of the nodes at 1 year are is α_1. The values of the X at the three nodes at the 1-year point are $+0.3464$, 0, and -0.3464.
- The corresponding values of $\ln S$ are $0.3464 + \alpha_1, \alpha_1,$ and $-0.3464 + \alpha_1$.
- The corresponding values of S are $e^{0.3464+\alpha_1}, e^{\alpha_1}, e^{-0.3464+\alpha_1}$, respectively. We require the expected value of S to equal the futures price, hence

$$0.1667e^{0.3464+\alpha_1} + 0.6666e^{\alpha_1} + 0.1667e^{-0.3464+\alpha_1} = 22$$

  The solution to this is $\alpha_1 = 3.071$. The values of S at the 1-year point are therefore, 30.49, 21.56, and 15.25.

- At the 2-year point, we first calculated the probabilities of nodes $E, F, G, H,$ and I being reached from the probabilities of nodes B, C and D being reached. The probability of reaching node F is the probability of reaching node B times the probability of moving from B to F plus the probability of reaching node C times the probability of moving from C to F. This is

$$0.1667 \times 0.6566 + 0.6666 \times 0.1667 = 0.2206.$$

  Likewise the probability of reaching node G is

$$0.1667 \times 0.2217 + 0.6666 \times 0.6566 + 0.1667 \times 0.2217 = 0.5183$$

  Similarly the probabilities of reaching nodes E, H and I are 0.0203, 0.2206, and 0.0203, respectively.

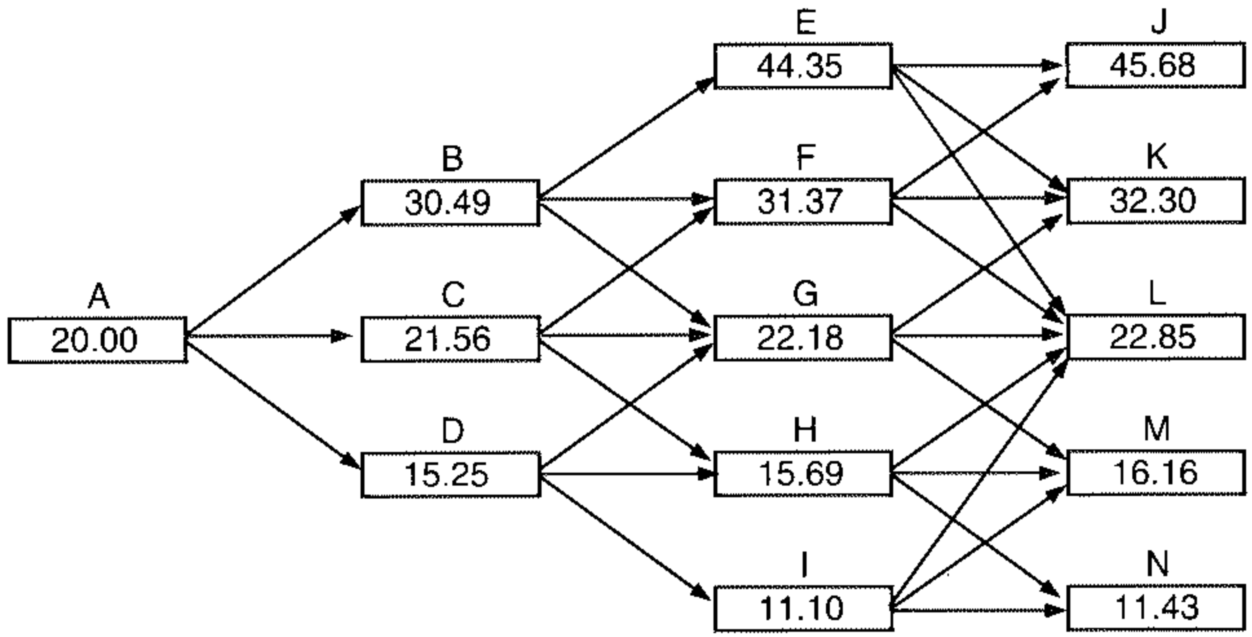
- The amount α_2 by which the nodes at time 2 years are displaced must satisfy

$$\begin{aligned} 0.0203e^{0.6928+\alpha_2} + 0.2206e^{0.3464+\alpha_2} + 0.5183e^{\alpha_2} \\ + 0.2206e^{-0.3464+\alpha_2} + 0.0203e^{-0.6928+\alpha_2} = 23 \end{aligned}$$

  The solution to this is $\alpha_2 = 3.099$. This means that the values of S at the 2-year point are 44.35, 31.37, 22.18, 15.69, and 11.10, respectively.

- A similar calculation can be carried out at time 3 years.

Figure 31.2 Tree for spot price of oil: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

- What do we do with this tree? We should think about "options" on a real financial objective and use such a binomial tree to value it.

Evaluating Options in an Investment Opportunity

Most investment projects involve options and can add value to a corporation. This can significantly alter the worth of a company. Some examples:

- **Abandonment options** This is an option to sell or close down a project. It is an American put option on the project's value. The strike price of the option is the liquidation (resale) value value of the project less any closing-down costs. When the liquidation value is low, the strike price can be negative. Abandonment options mitigate the impact of very poor investment outcomes and increase the initial valuation of a project.
- **Expansion options** This is an option to make further investments and increase the output if conditions are favorable. It is an American call option on the value of the additional capacity. It is an American call option on the value of additional capacity. The strike price of the call option is the cost of creating this additional capacity discounted to the time of option exercise. The strike price often depends on the initial investment. If management initially choose to build capacity in excess of the expected level of output, the strike price can be relatively small.
- **Contraction options** This is an option to reduce the scale of a project's operation. It is an American put option on the value of the lost capacity. The strike price is the present value of the future expenditures saved as seen at the time of exercise of the option.
- **Options to defer** One of the most important options open to a manager is the option to defer a project. This is an American call option on the value of the project.
- **Options to extend** Sometimes it is possible to extend the life of an asset by paying a fixed amount. This is a European call option on the asset's future value.

Options on an Investment Opportunity

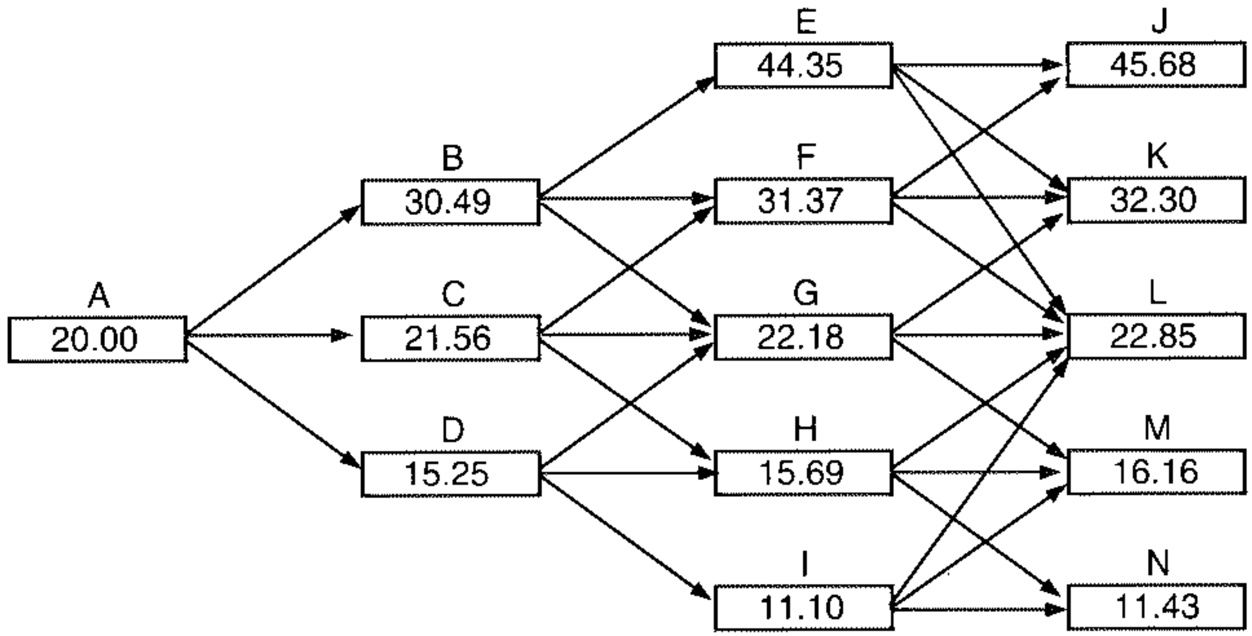
Consider an example of an investment with an embedded option. Suppose a company has to decide whether to invest \$15 million to obtain 6 million barrels of oil from a certain source at the rate of 2 million barrels per year for 3 years.

- The fixed costs of operating the equipment are \$6 million per year and the variable costs are \$17 per barrel. Assume that the risk-free interest rate is 10% per annum for al maturities, that the spot price of oil is \$20 per barrel, and that the 1-,2-, and 3-year futures prices are \$22, \$23, \$24 per barrel, respectively. We assume that the stochastic process for oil prices has been estimated with

$$d\ln S = [\theta(t) - a\ln S]\,dt + \sigma dz$$

  with $a = 0.1$ and $\sigma = 0.2$. Therefore we use our table

Figure 31.2 Tree for spot price of oil: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

which describes the behavior of oil prices in a risk-neutral world.

First assume that the project has no embedded options. The expected prices of oil in 1,2, and 3 years time in a risk neutral world is \$22, \$23, \$24, respectively.

The expected payoff from the project (in millions of dollars) in a risk-neutral world can be calculated from the cost data as 4.0, 6.0, and 8.0 in years 1,2,and 3 years, respectively. The value of the project is therefore,

$$-15.0 + 4.0e^{-0.1\times 1} + 6.0e^{-0.1\times 2} + 8.0e^{0.1\times 3} = -0.54$$

Therefore, the project should not be undertaken since it would reduce shareholder wealth by 0.54 million.

We can now try to recover the evaluation by using our tree. consider a node, say H.

- There is a 0.2217 chance that the price of oil at the end of the third year will be 22.85. Thus the third-year profit is

$$2 \times 22.85 - 2 \times \text{ variable } - \text{equipment} = 2 \times 22.85 - 2 \times 17 - 6 = 5.70.$$

- There is a 0.6566 chance that the price of oil at the end of the third year will be 16.16. Thus the third-year profit is

$$2 \times 16.16 - 2 \times 17 - 6 = -7.68.$$

- There is a 0.1217 chance that the price of oil at the end of the third year will be 11.43. Thus the third-year profit is

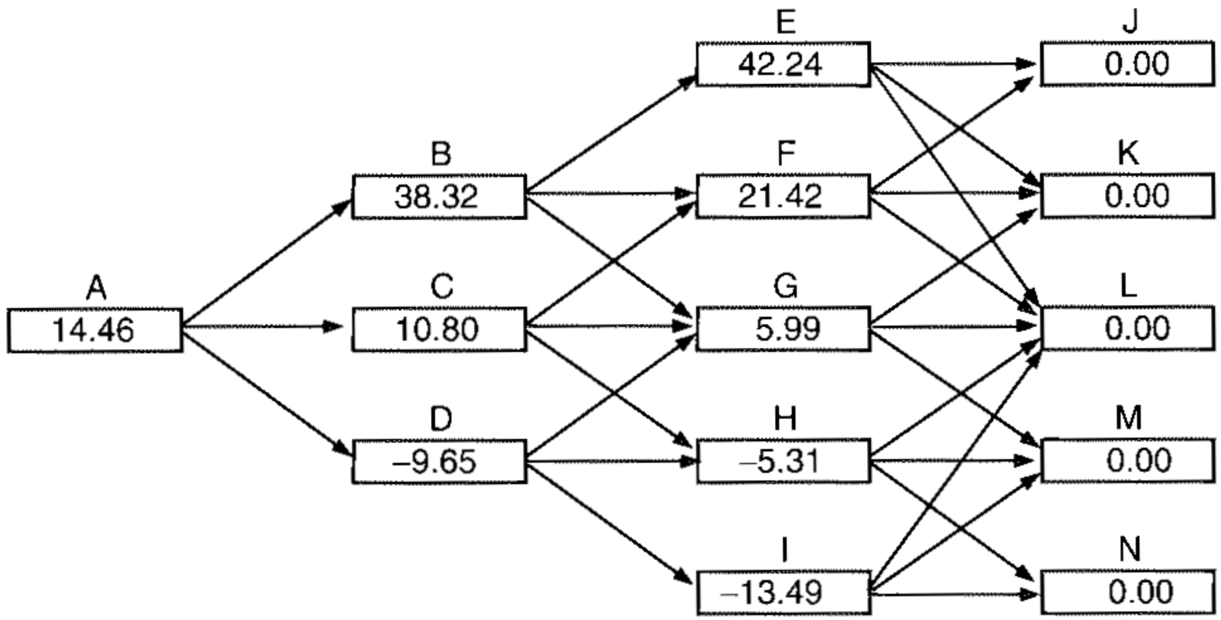
$$2 \times 11.43 - 2 \times 17 - 6 = -17.14.$$

- Therefore, the value at node H is

$$[0.2217 \times 5.70 + 0.6566 \times (-7.68) + 0.1217 \times (-17.14)]\, e^{-0.1\times 1} = -5.31$$

Continuing we find the tree

Figure 31.3 Valuation of base project with no embedded options: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

How do we get the value at nodes C for instance? There is a 0.1667 chance of moving to node F where the oil price is 31.37. The second year cash flow is then

$$2 \times 0.1667 - 2 \times 17 - 6 = 22.74$$

The value of subsequent cash flows at node F is 21.42. The total value of the project if we move to node F is therefore, $21.42 + 22.74 = 44.16$.

Similarly, the total value of the project if we move to nodes G and H are 10.35 and -13.93, respectively. The value of the project at node C is therefore,

$$[0.1667 \times 44.16 + 0.6666 \times 10.35 + 0.1667 \times (-13.93)]\, e^{-0.1 \times 1} = 10.80$$

The initial value of the project at node A is 14.46. When initial investment is taken into account the value of the project is therefore, -0.54. This agrees with earlier calculations.

Real Option on Gas

We now proceed with an option to abandon the project. Assume we can cancel our project at any time and that there is no salvage value and no further payments required once the project has been canceled.

Abandonment is an American put option with a strike price of zero and is value via a tree.

The put option should not be exercised at nodes E, F, and G because the value of the project is positive at these nodes.

It should be exercised at nodes H and I. The value of the put option is 5.31 and 13.49 at nodes H and I respectively. Rolling back through the tree, the value of the abandonment put option at node D if it is not exercised is

$$[0.1217 \times 13.49 + 0.6566 \times 5.31 + 0.2217 \times 0]\, e^{-0.1\times 1} = 4.64$$

The value of exercising the put option at node D is 9.65. This is greater than 4.64, and so the put should be exercised at node D. The value of the put option at node C is

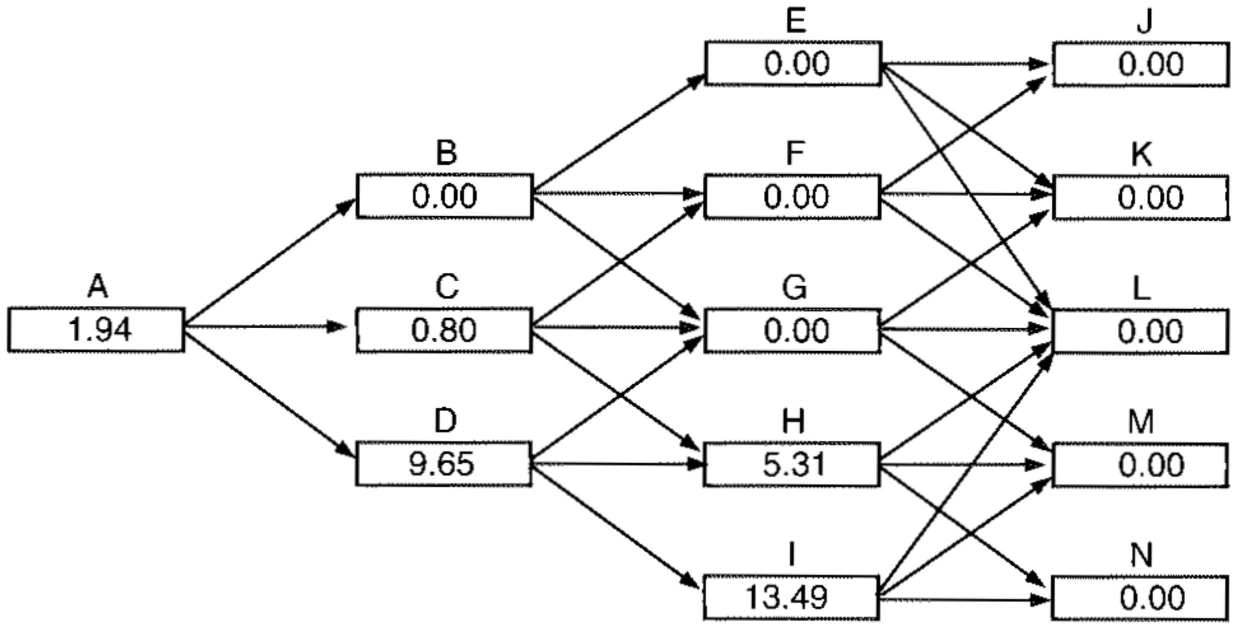
$$[0.1667 \times 0 + 0.6666 \times 0 + 0.1667 \times 5.31]\, e^{-0.1\times 1} = 0.80$$

and the value at node A is

$$[0.1667 \times 0 + 0.6666 \times 0.80 + 0.1667 \times 9.65]\, e^{-0.1\times 1} = 1.94$$

The abandonment option is therefore worth \$1.94 million. It increases the value of the project from -\$0.54 million to $+\$1.40$ million. A project that was previously unattractive now has a positive net value to shareholders.

Figure 31.4 Valuation of option to abandon the project: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



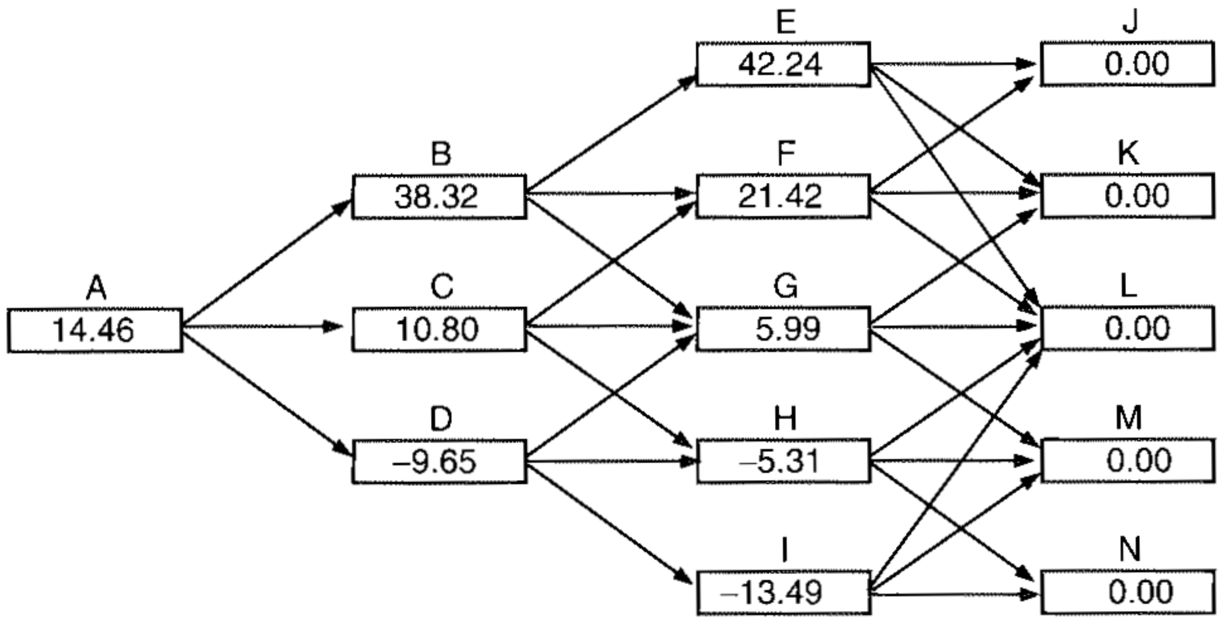
Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Expansion Option

Suppose next that the company has no abandonment option. Instead it has the option at any time to increase the scale of the project by 20%. The cost of doing this is $2 million.

Oil production increases from 2.0 to 2.4 million barrels. Variable costs remain $17 per barrel and fixed costs increase by 20% from $6 million to $7.2 million. This is an American call option to buy 20% of the base project in our tree for $2 million.

Figure 31.3 Valuation of base project with no embedded options: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



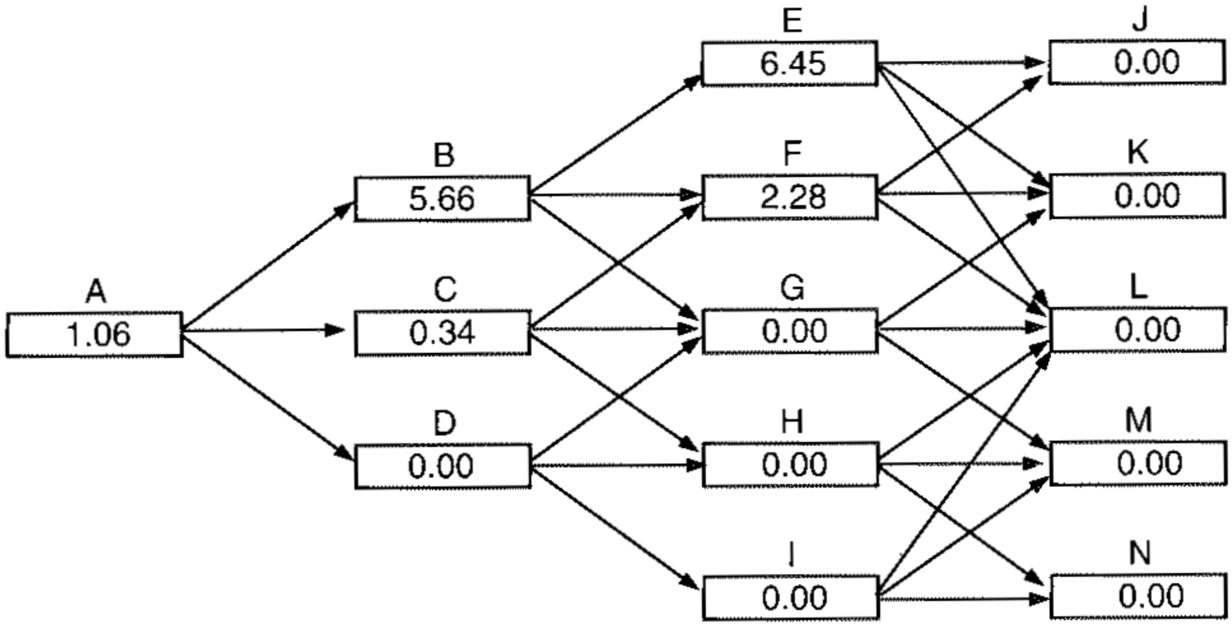
Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

At node E, the option should be exercised. The payoff is $0.2 \times 42.24 - 2 = 6.45$. At node F, it should also be exercised for a payoff of $0.2 \times 21.42 - 2 = 2.28$. At the nodes G, H, and I, the option should not be exercised. At node B, exercising is worth more than waiting and the option is worth $0.2 \times 38.32 - 2 = 5.66$. At node C, if the option is not exercised, it is worth

$$[0.1667 \times 2.28 + 0.6666 \times 0.00 + 0.1667 \times 0.00]\, e^{-0.1\times 1} = 0.34$$

If the option is exercised, it is worth $0.2 \times 10.80 - 2 = 0.16$. The option should therefore not be exercised at node C.

Figure 31.5 Valuation of option to expand the project: p_u, p_m, and p_d are the probabilities of "up", "middle", and "down" movements from a node.



Node:	A	B	C	D	E	F	G	H	I
p_u:	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m:	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d:	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

At node A, if not exercised, the option is worth

$$[0.1667 \times 5.66 + 0.6666 \times 0.34 + 0.1667 \times 0.00]\, e^{-0.1 \times 1} = 1.06$$

If the option is exercised it is worth $0.2 \times 14.46 - 2 = 0.89$. Early exercise is therefore not optimal at node A. In this case, the option increase the value of the project from -0.54 to $+0.52$. Again the project that was initially negative, has positive value.

The expansion option is relatively easy to value because one the option has been exercised, all subsequent cash inflows and outflows increase by 20%. In the case where fixed costs remain the same or increase by less than 20%, it is necessary to keep track of more information at the nodes in the price tree. In particular we would need:

1. The present value of subsequent fixed costs
2. The present value of subsequent revenues net of variable costs

The payoff from the option can then be calculated.

When a project has two or more options, they are typically not independent. The value of having both option A and option B is typically not the sum of the values of the two options.

To illustrate suppose that the company we have been considering has both abandonment and expansion options. The project cannot be expanded if it has already been abandoned. Moreover, the value of the put option to abandon depends on whether the project has been expanded. These two interactions between the options in our example can be handled by defining four states at each node:

1. Not already abandoned; not already expanded,
2. Not already abandoned; already expanded
3. Already abandoned, not already expanded
4. Already abandoned, already expanded

When we roll back through the tree, we calculate the combined value of the options at each node for all four alternatives. This approach to valuing path-dependent options is similar to what we did earlier.

Example 2

Suppose that the spot price, 6-month futures price, and 12-month futures price for wheat are 250, 260, and 270 cents per bushel, respectively. Suppose that the price of wheat follows the process

$$d\ln S = [\theta(t) - a\ln S]\,dt + \sigma dz$$

with $a = 0.05$ and $\sigma = 0.15$. Construct a two-time-step tree for the price of wheat in a risk-neutral world.

A farmer has a project that involves an expenditure of \$10,000 and a further expenditure of \$90,000 in 6 months. It will increase wheat that is harvested and sold by 40,000 bushels in 1 year.

what is the value of the project ? Suppose that the farmer can abandon the project in 6 months and avoid paying the \$90,000 cost at that time. What is the value of the abandonment option Assume a risk-free rate of 5% with continuous compounding.

Final Exam

Final Exam: May 7th in class.

- Forward & Futures, Swaps, Options and Risk-neutral valuation
- Arbitrage arguments & Put-Call parity
- Binomial Trees
- Stock Processes & Black-Scholes
- Greek Letters (Δ, Γ, $\mathcal{V}$)
- Volatility Smiles
- Numerical Methods: Bi-, Trinomial trees, Finite Difference Schemes, Monte-Carlo Methods
- Value At Risk
- Estimating Volatilities & Correlations
- Exotic Options (Asian, Barrier, Chooser)
- Martingales & Numeraires
- Black's Model, Caps, Swaptions