SUMS OF POWERS, STIRLING AND BERNOULLI NUMBERS

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1. Introduction

In your integral calculus class, you needed some explicit finite sums to evaluate definite integrals. For example,

$$\int_0^1 x \; dx \leftrightarrow 1 + 2 + 3 + \cdots + n = n(n+1)/2$$

$$\int_0^1 x^2 \; dx \leftrightarrow 1^2 + 2^2 + 3 + \cdots + n^2 = n(n+1)(2n+1)/6$$

You proved these idenities by induction on n.

But much more is true. In fact, if k is a positive integer, we will see that

$$1^k + 2^k + 3 + \cdots + n^k$$

is a polynomial in n of degree $k+1$, whose leading term is

$$\frac{n^{k+1}}{k+1}.$$

This leading term explains why

$$\int_0^1 x^k \; dx = \frac{1}{k+1}.$$

We will give two explicit formulas for this polynomial, using two sets of numbers: the Stirling numbers of the second kind, and Bernoulli numbers.

2. Stirling numbers

Definition 2.1. *A set partition* Π *of* $[n] = \{1, 2, \cdots, n\}$ *is an unordered decomposition of* $[n]$ *into disjioint sets, which are called blocks.*

Example 2.2. *Let* $n = 8$*, and* $\Pi = \{1,4,6\} \cup \{2,7\} \cup \{3\} \cup \{5,8\}$. $\{2,7\}$ *is a block of* Π. *This* π *has 4 blocks. We can write this in shorthand notation as* $146|27|3|58$.

Definition 2.3. *The number of set partitions of* $[n]$ *with exactly* k *blocks is the Stirling number of the* 2^{nd} *kind,* $S(n,k)$.

Example 2.4. $S(3,2) = 3$ *because the set partitions of* $[3]$ *with 2 blocks are*

$$1|23, \quad 12|3, \quad 13|2.$$

Date: September 17, 2019.

Exercise 2.5. *Find $S(4.2)$ and a general formula for $S(n, n-1)$ and $S(n, 2)$.*

Here are two recurrences for the Stirling numbers $S(n, k)$.

Proposition 2.6. *We have for $1 \leq k \leq n$*

$$S(n, k) = S(n-1, k-1) + k\ S(n-1, k).$$

Proof. Given any Π with exactly k blocks consider if the point n is in a block of Π by itself, or not, in one of the k blocks. These are the two terms on the right side. Note here that $S(n, 0) = 0$ for $n > 0$, and $S(0, 0) = 1$. □

Proposition 2.7. *We have for $1 \leq k \leq n$*

$$S(n, k) = \sum_{s=0}^{n-1} \binom{n-1}{s} S(n-1-s, k-1).$$

Proof. This time consider the block in which n appears. It has size $s+1$ for some integer s. Choosing the other s points in this block allows $\binom{n-1}{s}$ choices. What remains is a set partition with $k-1$ blocks on $n-1-s$ points. □

Definition 2.8. *The total number of set partitions of $[n]$ is the n^{th} Bell number,*

$$B_n = \sum_{k=1}^{n} S(n, k).$$

Exercise 2.9. *Find B_4.*

3. Bases of polynomials and Stirling numbers

Consider the vector space V_n of polynomials of degree at most n with complex coefficients. The usual basis for this space consists of the powers of x:

$$\{1, x, x^2, \cdots, x^n\}.$$

This basis is well adapted to calculus, the derivative maps one element to the previous one

$$\frac{d}{dx}(x^k) = kx^{k-1}.$$

For discrete situations, another basis is well adapted,

$$\{1, x, x(x-1), \cdots, x(x-1)\cdots(x-n+1)\}.$$

The difference operator

$$(\Delta f)(x) = f(x+1) - f(x)$$

again shifts these basis elements

$$(\Delta(x(x-1)\cdots(x-k+1)))(x) = kx(x-1)\cdots(x-k+2).$$

The Stirling numbers $S(n, k)$ gives the change of basis matrix for these two bases.

Proposition 3.1. *We have for* $n \geq 1$*,*

$$x^n = \sum_{k=1}^{n} S(n,k)x(x-1)\cdots(x-k+1)$$

Example 3.2.

$$\begin{aligned} x^3 =& S(3,1)x + S(3,2)x(x-1) + S(3,3)x(x-1)(x-2) \\ =& x + 3x(x-1) + x(x-1)(x-2). \end{aligned}$$

Proof. I'll give three proofs of this result, one which is combinatorial, the other two recursive (by induction.)

Proof #1: Let's prove this identity for all positive integers x. Since it is a polyomial identity over the complex numbers, it must be true for all x.

Let count all functions $f : [n] \to [x]$. Since each functional value has x choices, there are x^n such functions, which is the left side. For the right side, suppose the the image of f consists of k points. There are $\binom{x}{k}$ choices for this image. The preimage of f consists of an ordered set paritition of $[n]$ into k blocks, $S(n,k)k!$ choices, so

$$x^n = \sum_{k=1}^{n} S(n,k)\binom{x}{k}k!,$$

which is what we needed to prove.

Proof #2: Let's run an induction on n, $n = 1, 2$ is clear. Consider

$$\begin{aligned} &\sum_{k=1}^{n+1} S(n+1,k)x(x-1)\cdots(x-k+1) = \\ &\sum_{k=1}^{n+1}(S(n,k-1) + kS(n,k))x(x-1)\cdots(x-k+1) = \\ &\sum_{k=1}^{n} S(n,k)x(x-1)\cdots(x-k) + \sum_{k=1}^{n} kS(n,k)x(x-1)\cdots(x-k+1) = \\ &\sum_{k=1}^{n} S(n,k)(x-k+k)x(x-1)\cdots(x-k+1) = \\ &\quad x * x^n = x^{n+1}. \end{aligned}$$

Proof #3: Let's run an induction on n, this time using the second recursion for the Stirling numbers. Consider

$$\sum_{k=1}^{n+1} S(n+1,k)x(x-1)\cdots(x-k+1) =$$
$$\sum_{k=1}^{n+1}\sum_{s=0}^{n}\binom{n}{s}S(n-s,k-1)x(x-1)\cdots(x-k+1) =$$
$$\sum_{s=0}^{n}\binom{n}{s}\sum_{k=1}^{n}S(n-s,k-1)x(x-1)\cdots(x-k) =$$
$$\sum_{s=0}^{n}\binom{n}{s}x\sum_{k=1}^{n}S(n-s,k-1)(x-1)\cdots(x-k) =$$
$$\sum_{s=0}^{n}\binom{n}{s}x(x-1)^{n-s} =$$
$$x*x^n = x^{n+1}$$

where we have the binomial theorem.

□

The inverse change of basis matrix is given by signed Stirling numbers of the first kind.

4. Sums of powers

We now can state the polynomial formula for sums of powers using Stirling numbers of the second kind.

Theorem 4.1. *For a positive integer k,*

$$1^k + 2^k + \cdots + n^k = \sum_{j=1}^{k} S(k,j)\frac{(n+1)n*\cdots*(n-j+1)}{j+1}.$$

Example 4.2. *If $k=3$ this is*

$$\begin{aligned}1^3 + 2^3 + \cdots + n^3 &= S(3,1)(n+1)n/2 + S(3,2)(n+1)n(n-1)/3\\ &\quad + S(3,3)(n+1)n(n-1)(n-2)/4\\ &= (n+1)n/2 + 3(n+1)n(n-1)/3 + (n+1)n(n-1)(n-2)/4\\ &= n^2(n+1)^2/4.\end{aligned}$$

We will now prove Theorem 4.1. Let

$$S_k(n) = 1^k + 2^k + \cdots + n^k.$$

We need to solve the difference equation

$$S_k(n) - S_k(n-1) = n^k.$$

If the right side were a different polynomial in n of degree k, say

$$\binom{n}{k}$$

we would know a solution by Pascal's triangle, namely

$$A_k(n) = \binom{n+1}{k+1}$$

satisfies

$$A_k(n) - A_k(n-1) = \binom{n}{k}.$$

So we just expand n^k in terms of the binomial coefficients using Proposition 3.1, to get the solution

$$\begin{aligned} S_k(n) &= \sum_{j=1}^{k} S(k,j) j! \binom{n+1}{j+1} \\ &= \sum_{j=1}^{k} S(k,j) \frac{(n+1) * n * \cdots * (n-j+1)}{j+1} \end{aligned}$$

This sum may be expanded in the usual polynomial basis, n^j, using Bernoulli numbers.

Definition 4.3. *The Bernoulli numbers Ber_n are defined by*

$$\sum_{n=0}^{\infty} Ber_n \frac{z^n}{n!} = \frac{z}{e^z - 1} = Ber(z).$$

Example 4.4. *The first few values are*

$$Ber_0 = 1, \quad Ber_1 = -1/2, \quad Ber_2 = 1/6, \quad Ber_3 = 0, \quad Ber_4 = -1/30.$$

In fact all odd Bernoulli values are zero except for Ber_1.

Exercise 4.5. *Check that $Ber(z) + z/2$ is an odd function of z.*

It is somewhat more convenient to modify the Bernoulli numbers to $\hat{B}ern_n$ by changing the sign of a single term, $\hat{B}er_1 = 1/2$.

Theorem 4.6. *For a positive integer k,*

$$1^k + 2^k + \cdots + n^k = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} \hat{B}er_j n^{k+1-j}.$$

Note that the leading term is $n^{k+1}/(k+1)$, as promised.

Example 4.7. *If $k = 3$, this is*

$$\begin{aligned} 1^3 + 2^3 + \cdots + n^3 =& \frac{1}{4}(\hat{B}er_0 n^4 + 4\hat{B}er_1 n^3 + 6\hat{B}er_2 n^2 + 4\hat{B}er_3 n^1)) \\ =& \frac{1}{4}\left(n^4 + 2n^3 + n^2\right) = n^2(n+1)^2/4. \end{aligned}$$

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