1. Give examples of a finite ranked poset $P$ such that
(a) $P$ has the matching property but is not Sperner.
(b) $P$ is rank unimodal but not Sperner.
(c) $P$ is Sperner but not rank unimodal.
(d) $P$ is Sperner and rank unimodal, but does not have the matching property.

2. Prove that if $P$ is Sperner, and $P_{\max}$ is a maximum level, then the bipartite graphs
$$P_{\max-1} \cup P_{\max} \quad \text{and} \quad P_{\max+1} \cup P_{\max}$$
both have complete matchings.

3. Characterize all maximum sized antichains in the Boolean algebra $B_N$.

4. What is the Greene-Kleitman partition for the Boolean algebra $B_N$?

5. Can one prove log-concavity of the coefficients of the polynomial $\left[ \begin{array}{c} n \\ k \end{array} \right]_q$ using reality of the zeros?

6. Prove that $B_n(q)$ is Sperner by verifying that it is rank unimodal and has the matching property.

7. Here is another way to verify that $P = B_N(q)$ has the matching property. For $0 \leq k \leq N$ let $W_k$ be the $\mathbb{R}$ vector space whose basis is given by elements at level $k$ of $B_N(q)$, so $\dim(W_k) = \left[ \begin{array}{c} N \\ k \end{array} \right]_q$.
Let $D_k : W_k \to W_{k-1}$ and $U_k : W_k \to W_{k+1}$, $0 \leq k \leq N$, be the natural down and up linear transformations using the edges of $B_N(q)$.
(a) What is $D_{k+1}U_k - U_{k-1}D_k$ as a linear transformation on $W_k$?
(b) Show if $2k < n$, the map $U_k$ is 1-1, and find $\text{rank}(U_k)$.
(c) Show that the matrix of $U_k$ has a non-singular $\left[ \begin{array}{c} N \\ k \end{array} \right]_q \times \left[ \begin{array}{c} N \\ k \end{array} \right]_q$ submatrix, and conclude that a complete matching from $P_k$ to $P_{k+1}$ exists.

8. Let $\lambda_n = (n-1, n-2, \ldots, 1)$ be the “staircase” partition. Let $P_n = [\emptyset, \lambda_n]$ be the interval in Young’s lattice, namely the set of all partitions $\mu$ whose Ferrers diagram fit inside $\lambda_n$, under containment of Ferrers diagrams.
(a) Show that $|P_n| = C_n = \frac{1}{n+1} \binom{2n}{n}$, the $n^{th}$ Catalan number.
(b) If $R_n(q)$ is the rank generating function of $P_n$, find a version of $C_n = \sum_{k=1}^{n} C_{k-1}C_{n-k}, n \geq 1$, for $R_n(q)$.
(c) Is $P_n$ rank symmetric, rank unimodal*, or Sperner*?
(d) True or False?
$$\sum_{n=0}^{\infty} \frac{R_n(1/q)q^n (2^n)t^n}{(1-q)(1-q^2)\cdots(1-q^n)} = \sum_{n=0}^{\infty} \frac{(-t)^n q^{n^2}}{(1-q)(1-q^2)\cdots(1-q^n)} \sum_{n=0}^{\infty} \frac{(-t)^n q^{2n^2-n}}{(1-q)(1-q^2)\cdots(1-q^n)}$$
9. Let \( P_n = NC(n) \) the poset of non-crossing set partitions under refinement of blocks. Recall that \( |P_n| = C_n = \frac{1}{n+1} \binom{2n}{n} \), the \( n \)th Catalan number, and the \( k \)th level numbers are the Narayana numbers \( N_{n,k} = \frac{1}{k+1} \binom{n-1}{k} \binom{n}{k} \), \( 0 \leq k \leq n-1 \).

(a) Verify that \( P_n \) is a rank symmetric, rank unimodal poset.
(b) Verify that \( P_1, P_2, P_3, P_4 \) have symmetric chain decompositions by exhibiting one such decomposition on each Hasse diagram.
(c) Prove that \( P_n \) has a symmetric chain decomposition.

10. The inequality that we used for log-concavity
\[
e_k(x_1, \ldots, x_n)^2 \geq e_{k-1}(x_1, \ldots, x_n)e_{k+1}(x_1, \ldots, x_n), \quad 0 \leq k \leq n-1, \quad x_i > 0
\]
is a weaker version of the Newton inequalities
\[
\left( \frac{e_k(x_1, \ldots, x_n)}{\binom{n}{k}} \right)^2 \geq \left( \frac{e_{k-1}(x_1, \ldots, x_n)}{\binom{n}{k-1}} \right) \left( \frac{e_{k+1}(x_1, \ldots, x_n)}{\binom{n}{k+1}} \right), \quad 0 \leq k \leq n-1, \quad x_i > 0.
\]

(a) Take \( k = 1 \) and \( n = 3 \) and show that the Newton inequalities do not follow from termwise polynomial positivity.
(b) Prove the Newton inequalities by induction on \( n \), fixing \( k \). First verify the case \( n = k + 1 \) by showing a certain quadratic form is positive semidefinite. Then do the inductive case by assuming \( 0 < x_1 < x_2 < \cdots < x_n \) and letting
\[
P(t) = \prod_{i=1}^n (t + x_i), \quad P'(t) = n \prod_{i=1}^{n-1} (t + x'_i)
\]
where \( x_i < x'_i < x_{i+1} \). Use
\[
(n)e_k(x'_1, x'_2, \ldots, x'_{n-1}) = (n-k)e_k(x_1, \ldots, x_n), \quad 0 \leq k \leq n-1
\]
in the induction.

11. Let \( P \) be finite ranked poset and suppose that \( G \leq \text{Aut}(P) \). Define a poset \( P/G \) whose elements are the orbits \( O \) of \( G \) on \( P \), with order relation \( O_1 \leq O_2 \) if \( \exists \) there exists \( x \in O_1, y \in O_2 \), with \( x \leq y \) in \( P \). True or False: If \( P \) is Sperner, then \( P/G \) is Sperner.

12. In this problem you will prove the unimodality of the \( q \)-binomial coefficient by using an explicit formula, called the KOH identity.

First some notation. For an integer partition \( \lambda \), let \( |\lambda| \) be the sum of the parts of \( \lambda \). Let \( \lambda' \) be the conjugate of \( \lambda \), and let \( m_i(\lambda) \) be the multiplicity of the part \( i \) in \( \lambda \). For example, if \( \lambda = 544422111 \), then \( |\lambda| = 24, \lambda' = 96441 \), and \( m_4(\lambda) = 3 \). Finally, let
\[
n(\lambda) = \sum_i (i-1)\lambda_i = \sum_j \binom{\lambda'_j}{2}.
\]

It is
\[
\binom{N + k}{k}_q = \sum_{\lambda, |\lambda| = k} q^{2n(\lambda)} \prod_{i=1}^N \left( (N + 2)i - 2 \sum_{j=1}^{\lambda'_i} \lambda'_j + m_i(\lambda) \right) \binom{M}{3}_q.
\]

(a) Write out (KOH) for \( k = 3 \) and explain why it recursively proves that \( \binom{M}{3}_q \) is a unimodal polynomial in \( q \).
(b) Repeat (a) for a general \( k \) by showing that the individual terms in (KOH) are “centered” correctly.