Problem 8d:

Let \( \tilde{R}_n(q) = q^{(n)} R_n(1/q) \). Then part (b) which said

\[
R_n(q) = \sum_{k=1}^{n} q^{k(n-k)} R_{k-1}(q) R_{n-k}(q)
\]

becomes

(1) \[
\tilde{R}_n(q) = \sum_{k=1}^{n} q^{k-1} \tilde{R}_{k-1}(q) \tilde{R}_{n-k}(q).
\]

If we define the formal power series in \( t \)

(2) \[
A(t) = \sum_{n=0}^{\infty} \tilde{R}_n(q)t^n,
\]

then (1) is equivalent to

(3) \[
A(t) = 1 + tA(t)A(qt).
\]

Note that if \( q = 1 \) this is the Catalan generating function quadratic equation, \( A(t) = 1 + tA(t)^2 \).

Next, let

\[
B(t) = \sum_{n=0}^{\infty} \frac{q^{n^2-n}(-t)^n}{(1-q)(1-q^2)\cdots(1-q^n)}.
\]

Let’s find a \( q \)-difference equation for \( B(t) \). It is, upon subtracting term by term,

(4) \[
B(t) - B(tq) = -tB(tq^2).
\]

So

\[
\frac{B(tq)}{B(t)} = 1 + t \frac{B(tq^2)}{B(t)} \frac{B(tq)}{B(t)}
\]

which says that \( B(qt)/B(t) \) satisfies (3), so

\[
A(t) = B(tq)/B(t)
\]

is the first equation in Problem 8d.

If \( F(t) \) is the continued fraction, then we have \( F(t) = 1/(1 - tF(qt)) \) which is \( F(t) = 1 + tF(t)F(qt) \), again (3).
Note: It is possible to independently show that the continued fraction converges as a formal power series in \( t \) and is equal to \( A(t) \). Consider the more general finite continued fraction \((q^{i-1} = \lambda_i)\) which terminates at \( \lambda_n \), \( F_n(t, \lambda) \). For example

\[
F_4(t, \lambda) = \frac{1}{1 - \frac{t\lambda_1}{1 - \frac{t\lambda_2}{1 - \frac{t\lambda_3}{1 - t\lambda_4}}}}
\]

We have

\[
F_4(t, \lambda) = \frac{1}{1 - t\lambda_1 F_3(t, \lambda^+)} = \sum_{k=0}^{\infty} (t\lambda_1 F_3(t, \lambda^+))^k,
\]

where \( \lambda^+ \) means all of the \( \lambda \) indices have been increased by 1.

Let’s weight the down edge of a finite Dyck path from \( y \)-coordinate \( i \) to \( y \)-coordinate \( i - 1 \) by \( \lambda_i \). Then \( F_1(t, \lambda) = \sum_{k=0}^{\infty} (t\lambda_1)^k \) is the generating function for all Dyck paths which stay at or below the line \( y = 1 \), (only zigzags), and \( F_n(t, \lambda) = \frac{1}{1 - t\lambda_1 F_{n-1}(t, \lambda^+)} \) is the generating function for all Dyck paths which stay at or below the line \( y = n \). So for a fixed power of \( t \) say \( t^s \), once \( n \) is past \( s \), you get all such Dyck paths, and the coefficient of \( t^s \) in \( F_n(t) \) stabilizes as \( n \) increases. The infinite continued fraction

\[
\lim_{n \to \infty} F_n(t, \lambda)
\]

is the generating function for all Dyck paths with no restrictions on their heights.

By drawing a picture of a Dyck path one may see that the choice \( \lambda_i = q^{i-1} \) gives the polynomials \( \tilde{R}_n(q) \) as the coefficient of \( t^n \) in \( F(t, \lambda) = A(t) \).