Homework #3 Mathematics 8669 Due Monday April 4, 2016

1. Find the character table of the symmetric group $S_4$.

2. What is the orthogonality relation for the characters of the cyclic group of order $n$?

3. Show that for any character $\chi$ and $g \in G$, $|\chi(g)| \leq \chi(e) = \text{dimension}(\chi)$.

4. Let $G$ be a finite group of order $p^2$, $p$ a prime. By considering the possible dimensions of the irreducible representations of $G$, prove that $G$ is abelian.

5. Prove that the sum of any row of the character table of $G$ is a non-negative integer (see problem 9 for notation),

\[
\sum_{i=1}^{s} \chi^L(K_i) \text{ is a non-negative integer.}
\]

(Hint: Consider the character $\chi^G$ obtained by letting $G$ act on itself by conjugation. What is $\langle \chi^L, \chi^G \rangle$?)

6. The dihedral group $D_n$ is the group of order $2n$ which is the symmetry group of the regular $n$-gon. It may be given in terms of reflections and rotations as the set

\[D_n = \{ r^k : 0 \leq k \leq n - 1 \} \cup \{ sr^k : 0 \leq k \leq n - 1 \}\]

where $r^n = e$, $s^2 = e$, $srs = r^{-1}$. Show that the set of irreducible representations of $D_n$ consists of

(a) four 1-dimensional representations, and $(n/2 - 1)$ 2-dimensional representations for $n$ even
(b) two 1-dimensional representations, and $(n - 1)/2$ 2-dimensional representations for $n$ odd.

You should be able to construct these characters, using induced characters from the cyclic subgroup of order $n$.

7. Let $S_n$ act on the set $\{ 1, \cdots, n \}$ by the natural action of permutations. Let $V$ be the $n$-dimensional vector space over $\mathbb{C}$ whose basis is $\{ 1, \cdots, n \}$. In this problem you will prove that $V$ decomposes into two irreducibles: the identity of dimension 1, and another irreducible of dimension $n - 1$. Let $\chi^V$ denote the permutation character on $V$.

(a) Recall that the exponential formula says that

\[
\sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{g \in S_n} x^{\# \text{1-cycles of } g} = \exp \left( t x_1 + \sum_{k=2}^{\infty} \frac{t^k}{k} \right).
\]

Show that this formula implies that the average number of fixed points of $g \in S_n$ is 1 for $n \geq 1$, and that the average of $(\text{number of fixed points})^2$ is 2 for $n \geq 2$. 

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(b) Use part (a) to show that $\theta = \chi^V - \chi^{id}$ satisfies $< \theta, \theta > = 1$, and conclude that $\theta$ is irreducible.

8. Suppose that $V$ and $W$ are finite dimensional representations of $G$ over $\mathbb{C}$, with bases $\{v_1, \ldots, v_n\}$ and $\{w_1, \ldots, w_n\}$. Let $V \otimes W$ be the $mn$ dimensional $\mathbb{C}$-vector space whose basis is $v_i \otimes w_j$, $1 \leq i \leq n$, $1 \leq j \leq m$. Let $G$ act on $V \otimes W$ by

$$ \sum_{i,j} c_{ij} v_i \otimes w_j \rightsquigarrow \sum_{i,j} c_{ij} \rho_1(g)v_i \otimes \rho_2(g)w_j. $$

(a) Check that $\rho(g)$ is a non-singular linear transformation on $V \otimes W$, and $\rho(g_1g_2) = \rho(g_1)\rho(g_2)$.

(b) By explicitly computing traces, show that $\chi_{V \otimes W}(g) = \chi_V(g)\chi_W(g)$.

9. We proved in class that the center of the group algebra has dimension equal to the number of conjugacy classes of $G$. In this problem you will give two different bases for this center. Let $K_1, \ldots, K_s$ be the conjugacy classes of the finite group $G$. Let $\{\chi^L : L \text{ is irreducible}\}$ be the irreducible characters for $G$.

(a) Put

$$ \hat{K}_i = \sum_{g \in K_i} g \in C[G], \quad 1 \leq i \leq s. $$

Show that $g\hat{K}_i = \hat{K}_ig$ for all $g \in G$, and conclude that $\{\hat{K}_1, \ldots, \hat{K}_s\}$ is a basis for the center of $C[G]$.

(b) For an irreducible $L$ of $G$ put

$$ e_L = \frac{\dim(L)}{|G|} \sum_{g \in G} \chi^L(g^{-1})g \in C[G]. $$

Show that $ge_L = e_Lg$ for all $g \in G$, and conclude that $\{e_L : L \text{ is irreducible}\}$ is a basis for the center of $C[G]$.

(c) Using the orthogonality relation for matrix elements, show that $e_Ke_L = e_K\delta_{KL}$ and find $\sum_K e_K$.

(d) For each irreducible $L$, define a map $\phi_L : Center(C[G]) \to \mathbb{C}$ by

$$ \phi_L(z) = \frac{1}{\dim(L)} \sum_{g \in G} z(g)\chi^L(g). $$

Show that $\phi_L$ is an algebra homomorphism by checking that $\phi_L(e_K) = \delta_{KL}$ and using part (c).

10. In the notation of Problem 9,
(a) prove that there exists non-negative integers $\alpha^k_{ij}$ such that
\[ \hat{K}_i \hat{K}_j = \sum_{k=1}^{s} \alpha^k_{ij} \hat{K}_k. \]

(b) prove that there exists non-negative integers $\beta^K_{LJ}$ such that
\[ \chi^L(g) \chi^J(g) = \sum_{K} \beta^K_{LJ} \chi^K(g). \]

11. You may use the following fact from algebra: If $R$ is a commutative ring and $x \in R$, then $x$ is integral over $\mathbb{Z}$ if, and only if, the subring $\mathbb{Z}[x]$ of $R$ generated by $x$ is finitely generated, if, and only if, $R$ contains a finitely generated $\mathbb{Z}$-submodule which contains $\mathbb{Z}[x]$. Prove that $\hat{K}_i$ is integral over $\mathbb{Z}$, that is $\hat{K}_i$ satisfies a polynomial equation with integral coefficients whose leading term has coefficient 1.

12. This problem uses problems 9 and 11 to show $\text{dim}(L)$ divides $|G|$ for an irreducible $L$.

(a) Let $z \in \text{Center}(\mathbb{C}[G])$, so that $z$ may be considered as a function on the conjugacy classes $K_i$. If $z(K_i)$ is integral over $\mathbb{Z}$ for $1 \leq i \leq s$, verify that
\[ X = \sum_{g \in G} z(g)g = \sum_{i=1}^{s} z(K_i)K_i \]
is also integral over $\mathbb{Z}$.

(b) Since $\phi_L$ is an algebra homomorphism, conclude that $\phi_L(X)$ is also integral over $\mathbb{Z}$.

(c) Finally prove that $\text{dim}(L)$ divides $|G|$ by choosing $z(K_i) = \chi^L(g^{-1})$, and noting that
\[ \phi_L(X) = \frac{|G|}{\text{dim}(L)} < X_L, X_L > = \frac{|G|}{\text{dim}(L)} \]
is integral over $\mathbb{Z}$. 