2. Prove that if $P$ is Sperner, and $P_{\text{max}}$ is a maximum level, then the bipartite graphs

$$P_{\text{max}-1} \cup P_{\text{max}} \quad \text{and} \quad P_{\text{max}+1} \cup P_{\text{max}}$$

both have complete matchings.

**Solution:** Suppose, by contradiction, that there is no complete match from $P_{\text{max}-1} \rightarrow P_{\text{max}}$. Then by Hall’s theorem there exists a subset $S \subset P_{\text{max}-1}$ whose relatives $R(S) \subset P_{\text{max}}$ satisfy $|S| > |R(S)|$. Then $A = S \cup (P_{\text{max}} - R(S))$ is an antichain of size larger than $P_{\text{max}}$, which is a contradiction.

3. Characterize all maximum sized antichains in the Boolean algebra $B_N$.

**Solution:** Claim: The maximum sized antichains are precisely the maximum sized level sets, and no others.

As in lecture, the LYM property for $B_N$ implies that a maximum sized antichain must lie inside the maximum levels. So for $N$ even this is unique. Let’s assume $N = 2m + 1$ is odd, and prove that a maximum sized antichain $A$ could not be in both levels, $A = A_1 \cup A_2$, $\emptyset \neq A_1 \subset B_N(m)$, $\emptyset \neq A_2 \subset B_N(m+1)$ is impossible.

Note that the bipartite graph $G = B_N(m) \cup B_N(m+1)$ is regular of degree $m+1$. Let $R(A_1) \subset B_N(m+1)$ be the relatives of $A_1$. Because we know that a complete match exists in $G$, by Hall’s condition $|A_1| \leq |R(A_1)|$. But since $A_2 \subset B_N(m+1) - R(A_1)$ and $|A_1| + |A_2| = \binom{2m+1}{m}$, we have $|A_1| = |R(A_1)|$, so each of the $|A_1| |A_1|$ edges from $A_1$ go to $R(A_1)$, and each of the $(m+1)|A_1|$ edges from $R(A_1)$ do in fact go to $A_1$. The same reasoning applies to $A_2$ and $R(A_2)$. So the bipartite graph $G$ is disconnected, which is a contradiction.

7. Here is another way to verify that $P = B_N(q)$ has the matching property. For $0 \leq k \leq N$ let $W_k$ be the $\mathbb{R}$ vector space whose basis is given by elements at level $k$ of $B_N(q)$, so $\dim(W_k) = \begin{bmatrix} N \\ k \end{bmatrix}_q$.

Let $D_k : W_k \to W_{k-1}$ and $U_k : W_k \to W_{k+1}$, $0 \leq k \leq N$, be the natural down and up linear transformations using the edges of $B_N(q)$.

(a) What is $D_{k+1}U_k - U_{k-1}D_k$ as a linear transformation on $W_k$?

(b) Show if $2k < n$, the map $U_k$ is 1-1, and find $\text{rank}(U_k)$.

**Solution:** From (a) $D_{k+1}U_k = U_{k-1}D_k + c_k I$, where $c_k > 0$. As a amtrix $U_{k-1} = D_k^T$, so $U_{k-1}D_k$ is positvie semidefine, therefore $D_{k+1}U_k$ is positive definite, so invertible, This implies that $\ker(U_k) = \emptyset$ and $U_k$ is injective and $\text{rank}(U_k) = \begin{bmatrix} N \\ k \end{bmatrix}_q$.

(c) Show that the matrix of $U_k$ has a non-singular $\begin{bmatrix} N \\ k \end{bmatrix}_q \times \begin{bmatrix} N \\ k \end{bmatrix}_q$ submatrix, and conclude that a complete matching from $P_k$ to $P_{k+1}$ exists.

**Solution:** Any $m \times n$ matrix $A$ with $\text{rank}(A) = m$ has an $m \times m$ non-singular matrix $B$, by choosing $m$ linearly independent columns. Here we have

$$\det(B) = \sum_{\pi \in S_m} \text{sign}(\pi) \prod_{i=1}^{m} B_{\pi(i)},$$

and $\det(B) \neq 0$ implies that $B_{\pi(i)} \neq 0$ for all $i$ for some $\pi \in S_m$.

Applying this to part (b), the permutation $\pi$ gives the matching.
9. Let $P_n = NC(n)$ the poset of non-crossing set partitions under refinement of blocks. Recall that $|P_n| = C_n = \frac{1}{n+1} \binom{2n}{n}$, the $n$th Catalan number, and the $k$th level numbers are the Narayana numbers $N_{n,k} = \frac{1}{k+1} \binom{n-1}{k} \binom{n}{k}$, $0 \leq k \leq n-1$.

(c) Prove that $P_n$ has a symmetric chain decomposition.

**Solution:**

Let’s do this by induction on $n$, the first few cases were done in part (b). Since $\text{rank}(P_n) = n-1$, we need saturated chains whose bottom and top ranks add to $n-1$.

The main idea is to consider the block containing 1. Suppose the next smallest element in 1’s block

Next we prove the Newton inequalities by induction on $n$.

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Finally we deal with the two remaining cases: 1 in a block by itself or 12 in a block. These are 

10. The inequality that we used for log-concavity

$$e_k(x_1, \ldots, x_n)^2 \geq e_{k-1}(x_1, \ldots, x_n)e_{k+1}(x_1, \ldots, x_n), \quad 0 \leq k \leq n-1, \quad x_i > 0$$

is a weaker version of the Newton inequalities

$$\left(\frac{e_k(x_1, \ldots, x_n)}{\binom{n}{k}}\right)^2 \geq \left(\frac{e_{k-1}(x_1, \ldots, x_n)}{\binom{n}{k-1}}\right) \left(\frac{e_{k+1}(x_1, \ldots, x_n)}{\binom{n}{k+1}}\right), \quad 0 \leq k \leq n-1, \quad x_i > 0.$$
interlace with the zeros of $P(t)$, so we can write

$$P'(t) = n \prod_{i=1}^{n-1} (t + x'_i), \quad x_i < x'_i < x_{i+1}, \quad 1 \leq i \leq n - 1.$$  

Finding the coefficient of $t^{n-1-k}$ in $P'(t)$ gives

$$(n) e_k(x'_1, x'_2, \ldots, x'_{n-1}) = (n-k)e_k(x_1, \ldots, x_n) \quad 0 \leq k \leq n - 1.$$  

So by induction

$$\left( \frac{e_k(x_1, \ldots, x_n)}{\binom{n}{k}} \right)^2 = \left( \frac{e_k(x'_1, \ldots, x'_{n-1})}{\binom{n-1}{k}} \right)^2$$

$$\geq \left( \frac{e_{k-1}(x'_1, \ldots, x'_{n-1})}{\binom{n-1}{k-1}} \right) \left( \frac{e_{k+1}(x'_1, \ldots, x'_{n-1})}{\binom{n-1}{k+1}} \right)$$

$$= \left( \frac{e_{k-1}(x_1, \ldots, x_n)}{\binom{n}{k-1}} \right) \left( \frac{e_{k+1}(x_1, \ldots, x_n)}{\binom{n}{k+1}} \right).$$

12. In this problem you will prove the unimodality of the $q$-binomial coefficient by finding an explicit formula, called the KOH identity.

First some notation. For an integer partition $\lambda$, let $|\lambda|$ be the sum of the parts of $\lambda$. Let $\lambda'$ be the conjugate of $\lambda$, and let $m_i(\lambda)$ be the multiplicity of the part $i$ in $\lambda$. For example, if $\lambda = 544422111$, then $|\lambda| = 24$, $\lambda' = 96441$, and $m_1(\lambda) = 3$. Finally, let

$$n(\lambda) = \sum_i (i-1)\lambda_i = \sum_j \binom{\lambda_j}{2}.$$  

It is

$$(\text{KOH}) \quad \left[ \begin{array}{c} N+k \\ k \end{array} \right]_q = \sum_{\lambda: |\lambda| = k} q^{2n(\lambda)} \prod_{i=1}^{\infty} \left( N+2i - 2 \sum_{j=1}^{\lambda'} \lambda_j + m_i(\lambda) \right)_q.$$  

(a) Write out (KOH) for $k = 3$ and explain why it recursively proves that $\left[ \begin{array}{c} M \\ 3 \end{array} \right]_q$ is a unimodal polynomial in $q$.

**Solution:** Since $k = 3$ there are 3 partitions in the sum on the right side $\lambda = 3, 21, 111$. The (KOH) identity becomes

$$\left[ \begin{array}{c} N+3 \\ 3 \end{array} \right]_q = \left[ \begin{array}{c} 3N+1 \\ 1 \end{array} \right]_q + q^2 \left[ \begin{array}{c} N-1 \\ 1 \end{array} \right]_q \left[ \begin{array}{c} 2N-1 \\ 1 \end{array} \right]_q + q^6 \left[ \begin{array}{c} N-1 \\ 3 \end{array} \right]_q.$$  

Now suppose we try to prove that $\left[ \begin{array}{c} M \\ 3 \end{array} \right]_q$ is unimodal by induction on $M$. If we can show that each of the three terms in (1) is unimodal and centered at the same center as $\left[ \begin{array}{c} N+3 \\ 3 \end{array} \right]_q$, which is $3N/2$, we are done. Since the second term is a product of symmetric unimodal polynomials, it is certainly symmetric and unimodal, as are the first and last (by induction) terms.

1. $\left[ \begin{array}{c} 3N+1 \\ 1 \end{array} \right]_q$: smallest term $q^0$, largest term $q^{3N}$, $0 + 3N = 3N$ works.

2. $q^2 \left[ \begin{array}{c} N-1 \\ 1 \end{array} \right]_q \left[ \begin{array}{c} 2N-1 \\ 1 \end{array} \right]_q$: smallest term $q^2$, largest term $q^{2+(N-2)+(2N-2)}$, $2 + 3N - 2 = 3N$ works.
(3) $q^6 \left[ \frac{N-1}{3} \right]_q$: smallest term $q^6$, largest term $q^{6+3(N-4)}$, $6 + 3N - 6 = 3N$ works.

(b) Repeat (a) for a general $k$ by showing that the individual terms in (KOH) are “centered” correctly.

**Solution:** The induction goes through as before, we must check the centering condition for each term. This is

$$2n(\lambda) + \left( 2n(\lambda) + \sum_{i=1}^{\infty} m_i(\lambda)((N+2)i - 2 \sum_{j=1}^{i} \lambda'_i) \right) = kN.$$ 

Since

$$\sum_{i=1}^{\infty} m_i(\lambda)i = k$$

we must show that

$$2n(\lambda) + k = \sum_{i=1}^{\infty} m_i(\lambda) \sum_{j=1}^{i} \lambda'_i.$$ 

Here is an example how this is proven, the general case is the same.

Let $\lambda = 322111$, so $k = 10$, $n(\lambda) = 18$. Let compute $n(\lambda) + n(\lambda) + k$ pictorially:

```
0 0 0 5 2 0 1 1 1
1 1 4 1 1 1
2 2 3 0 1 1
3 2
4 1
5
```

Adding these we find

```
6 3 1
6 3
6 3
6
6
6
```

which is the right side of (2).