Thanks

Research Conducted by the Speaker with

Sun Kim
Ohio State University

and

Alexandru Zaharescu
University of Illinois at Urbana-Champaign
I have shown you today the highest secret of my own realization. It is supreme and most mysterious indeed.

Verse 575, Vivekachudamani of Adi Shankaracharya
Sixth Century, A.D.
\[ d(n) \] denotes the number of positive divisors of \( n \).
The Dirichlet Divisor Problem

$d(n)$ denotes the number of positive divisors of $n$.

Theorem (Dirichlet, 1849)

For $x > 0$, set

$$D(x) := \sum'_{n \leq x} d(n) = x(\log x + 2\gamma - 1) + \frac{1}{4} + \Delta(x), \quad (1)$$

where the prime on the summation sign on the left-hand side indicates that if $x$ is an integer then only $\frac{1}{2}d(x)$ is counted, $\gamma$ is Euler’s constant, and $\Delta(x)$ is the “error term.” Then, as $x \to \infty$,

$$\Delta(x) = O(\sqrt{x}). \quad (2)$$
The Dirichlet Divisor Problem

Figure: The Dirichlet Divisor Problem
**Conjecture** For each $\epsilon > 0$, as $x \to \infty$,

$$\Delta(x) = O(x^{1/4+\epsilon}).$$
The Dirichlet Divisor Problem

**Conjecture** For each $\epsilon > 0$, as $x \to \infty$,

$$\Delta(x) = O(x^{1/4+\epsilon}).$$

**Theorem (Hardy, 1916)**

$$\Delta(x) = \Omega_+ \left( \{x \log x\}^{1/4} \log \log x \right).$$
The Dirichlet Divisor Problem

**Conjecture** For each $\epsilon > 0$, as $x \to \infty$,

$$\Delta(x) = O(x^{1/4+\epsilon}).$$

**Theorem (Hardy, 1916)**

$$\Delta(x) = \Omega_+ (\{x \log x\}^{1/4} \log \log x).$$

**Theorem (Voronoï, 1904; Huxley, 2003)**

As $x \to \infty$,

$$\Delta(x) = O(x^{1/3} \log x),$$

$$= O(x^{131/416+\epsilon})$$
**The Dirichlet Divisor Problem**

**Conjecture** For each $\epsilon > 0$, as $x \to \infty$, 

$$\Delta(x) = O(x^{1/4+\epsilon}).$$

**Theorem (Hardy, 1916)**

$$\Delta(x) = \Omega_+ (\{x \log x\}^{1/4} \log \log x).$$

**Theorem (Voronoi, 1904; Huxley, 2003)**

As $x \to \infty$,

$$\Delta(x) = O(x^{1/3} \log x),
\quad = O(x^{131/416+\epsilon}),$$

$$\frac{131}{416} = 0.3149 \ldots$$
The Dirichlet Divisor Problem

Theorem (Voronoï, 1904)

If $x > 0$,

$$
\sum_{n \leq x}^\prime d(n) = x (\log x + 2\gamma - 1) + \frac{1}{4} + \sum_{n=1}^\infty d(n) \left( \frac{x}{n} \right)^{1/2} l_1(4\pi \sqrt{nx}),
$$

where $l_1(z)$ is defined by

$$
l_\nu(z) := -Y_\nu(z) - \frac{2}{\pi} K_\nu(z).
$$
\[ a < \theta < 1. \]

\[
\left[ \frac{\pi}{2} \right] \sin \pi \theta + \left[ \frac{\pi}{2} \right] \sin 2 \pi \theta + \left[ \frac{\pi}{2} \right] \sin 3 \pi \theta + \left[ \frac{\pi}{2} \right] \sin 4 \pi \theta + \left[ \frac{\pi}{2} \right] \sin 5 \pi \theta + \left[ \frac{\pi}{2} \right] \sin 6 \pi \theta + \left[ \frac{\pi}{2} \right] \sin 7 \pi \theta + \left[ \frac{\pi}{2} \right] \sin 8 \pi \theta + 
\]

\[
= \pi x \left( \frac{1}{2} - \theta \right) - \frac{1}{2} \cos \pi \theta + \frac{1}{2} \sqrt{x} \sum_{\lambda=1}^{\infty} \left\{ \frac{J_1(4 \pi \sqrt{\lambda \theta x})}{\sqrt{\lambda \theta}} - \frac{J_1(4 \pi \sqrt{\lambda (1-\theta) x})}{\sqrt{\lambda (1-\theta)}} + \frac{J_1(4 \pi \sqrt{\lambda (2-\theta) x})}{\sqrt{\lambda (2-\theta)}} - \frac{J_1(4 \pi \sqrt{\lambda (3-\theta) x})}{\sqrt{\lambda (3-\theta)}} + \cdots \right\}
\]

where \( \left[ \frac{x}{2} \right] \) denotes the greatest integer in \( x \) if \( x \) is not an integer and \( x - \frac{1}{2} \) if \( x \) is an integer.

\[
\left[ \frac{\pi}{2} \right] \cos 2 \pi \theta + \left[ \frac{\pi}{2} \right] \cos 4 \pi \theta + \left[ \frac{\pi}{2} \right] \cos 6 \pi \theta + \left[ \frac{\pi}{2} \right] \cos 8 \pi \theta + \cdots
\]

\[
= \frac{1}{2} \sqrt{x} \sum_{\lambda=1}^{\infty} \left\{ \frac{T_1(4 \pi \sqrt{\lambda \theta x})}{\sqrt{\lambda \theta}} - \frac{T_1(4 \pi \sqrt{\lambda (1-\theta) x})}{\sqrt{\lambda (1-\theta)}} + \frac{T_1(4 \pi \sqrt{\lambda (2-\theta) x})}{\sqrt{\lambda (2-\theta)}} - \frac{T_1(4 \pi \sqrt{\lambda (3-\theta) x})}{\sqrt{\lambda (3-\theta)}} + \cdots \right\}
\]

where

\[ T_1(x) = \Gamma(x) - \gamma(x). \]
Define

\[ F(x) = \begin{cases} 
  [x], & \text{if } x \text{ is not an integer}, \\
  x - \frac{1}{2}, & \text{if } x \text{ is an integer}, 
\end{cases} \tag{4} \]

where, as customary, \([x]\) is the greatest integer less than or equal to \(x\).
Define

\[ F(x) = \begin{cases} 
[x], & \text{if } x \text{ is not an integer}, \\
 x - \frac{1}{2}, & \text{if } x \text{ is an integer}, 
\end{cases} \] (4)

where, as customary, \([x]\) is the greatest integer less than or equal to \(x\).

\[ I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi} K_\nu(z). \] (5)
Let $F(x)$ be defined by (4), and let $I_1(x)$ be defined by (5). For $x > 0$ and $0 < \theta < 1$,

$$
\sum_{n=1}^{\infty} F \left( \frac{x}{n} \right) \cos(2\pi n\theta) = \frac{1}{4} - x \log(2 \sin(\pi\theta))
$$

$$
+ \frac{1}{2} \sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{I_1 \left( 4\pi \sqrt{m(n+\theta)x} \right)}{\sqrt{m(n+\theta)}} + \frac{I_1 \left( 4\pi \sqrt{m(n+1-\theta)x} \right)}{\sqrt{m(n+1-\theta)}} \right\}.
$$
As \( x \to \infty \),

\[
Y_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\pi \nu}{2} - \frac{\pi}{4} \right) + O \left( \frac{1}{x^{3/2}} \right),
\]

\[
K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} + O \left( e^{-x} \frac{1}{x^{3/2}} \right).
\]
As $x \to \infty$,

$$Y_\nu(x) = \sqrt{\frac{2}{\pi x}} \sin \left(x - \frac{\pi \nu}{2} - \frac{\pi}{4}\right) + O \left(\frac{1}{x^{3/2}}\right),$$

$$K_\nu(x) = \sqrt{\frac{\pi}{2x}} e^{-x} + O \left(e^{-x} \frac{1}{x^{3/2}}\right).$$
Fix $x > 0$ and set $\theta = u + \frac{1}{2}$, where $-\frac{1}{2} < u < \frac{1}{2}$. Recall that $F(x)$ is defined in (4). If the identity below is valid for at least one value of $\theta$, then it exists for all values of $\theta$, and

$$\sum_{1 \leq n \leq x} (-1)^n F \left( \frac{x}{n} \right) \cos(2\pi nu) - \frac{1}{4} + x \log(2 \cos(\pi u))$$

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2} + u} \lim_{M \to \infty} \left\{ \sum_{m=1}^{M} \sin \left( \frac{2\pi(n + \frac{1}{2} + u)x}{m} \right) \right\}$$

$$- \int_{0}^{M} \sin \left( \frac{2\pi(n + \frac{1}{2} + u)x}{t} \right) dt$$
Theorem (Continued)

\[ + \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2} - u} \lim_{M \to \infty} \left\{ \sum_{m=1}^{M} \sin \left( \frac{2\pi(n + \frac{1}{2} - u)x}{m} \right) \right\} \]

\[- \int_{0}^{M} \sin \left( \frac{2\pi(n + \frac{1}{2} - u)x}{t} \right) dt \} . \]

Moreover, the series on the right-hand side of (6) converges uniformly on compact subintervals of \((-\frac{1}{2}, \frac{1}{2})\).
Another Interpretation of Ramanujan’s Claim

Entry

Let $F(x)$ be defined by (4), and let $l_1(x)$ be defined by (5). For $x > 0$ and $0 < \theta < 1$,

$$
\sum_{n=1}^{\infty} F \left( \frac{x}{n} \right) \cos(2\pi n\theta) = \frac{1}{4} - x \log(2 \sin(\pi \theta))
$$

$$
+ \frac{1}{2^2} \sqrt{x} \sum_{m \geq 1, n \geq 0} \left\{ l_1 \left( \frac{4\pi \sqrt{m(n + \theta)x}}{\sqrt{m(n + \theta)}} \right) + l_1 \left( \frac{4\pi \sqrt{m(n + 1 - \theta)x}}{\sqrt{m(n + 1 - \theta)}} \right) \right\} .
$$
If $\chi$ denotes any character modulo $q$, we define

$$d_\chi(n) := \sum_{d|n} \chi(d),$$
If $\chi$ denotes any character modulo $q$, we define

$$d_\chi(n) := \sum_{d | n} \chi(d),$$

$$\zeta(s)L(s, \chi) = \sum_{n=1}^{\infty} d_\chi(n)n^{-s},$$
If \( \chi \) denotes any character modulo \( q \), we define

\[
d_{\chi}(n) := \sum_{d \mid n} \chi(d),
\]

\[
\zeta(s)L(s, \chi) = \sum_{n=1}^{\infty} d_{\chi}(n)n^{-s},
\]

\[
\tau(\chi) := \sum_{h=1}^{q-1} \chi(h)e^{2\pi ih/q}.
\]
Functional Equation for Nonprincipal Even Primitive Characters

\[
\left( \frac{\pi}{\sqrt{q}} \right)^{-2s} \Gamma^2(s) \zeta(2s) L(2s, \chi)
\]

\[
= \frac{\tau(\chi)}{\sqrt{q}} \left( \frac{\pi}{\sqrt{q}} \right)^{-2 \left( \frac{1}{2} - s \right)} \Gamma^2 \left( \frac{1}{2} - s \right) \zeta(1 - 2s) L(1 - 2s, \overline{\chi}).
\]
If $\chi$ is a nonprincipal even primitive character modulo $q$, then

$$\sum_{n \leq x}' d_\chi(n) = \frac{\sqrt{q}}{\tau(\chi)} \sum_{n=1}^{\infty} d_{\overline{\chi}}(n) \sqrt{\frac{x}{n}} I_1 \left(4\pi \sqrt{nx/q}\right)$$

$$- \frac{x}{\tau(\chi)} \sum_{h=1}^{q-1} \overline{\chi}(h) \log \left(2 \sin\left(\frac{\pi h}{q}\right)\right).$$  (7)
• We have proved Ramanujan’s Second Identity with the order of summation reversed.
• We have proved Ramanujan’s Second Identity with the order of summation reversed.
• We have proved Ramanujan’s Second Identity with the product of the summation indices tending to infinity.
Summary

- We have proved Ramanujan’s Second Identity with the order of summation reversed.
- We have proved Ramanujan’s Second Identity with the product of the summation indices tending to infinity.
- We have not proved Ramanujan’s Second Identity with the order of summation as written by Ramanujan.
- We have proved Ramanujan’s First Identity under all three interpretations.

Road Blocks: The plus sign between the two Bessel functions; singularities at 0.

The proofs under different interpretations are completely different.

New methods of estimating trigonometric sums are introduced.
Summary

- We have proved Ramanujan’s Second Identity with the order of summation reversed.
- We have proved Ramanujan’s Second Identity with the product of the summation indices tending to infinity.
- We have not proved Ramanujan’s Second Identity with the order of summation as written by Ramanujan.
- We have proved Ramanujan’s First Identity under all three interpretations.
• We have proved Ramanujan’s Second Identity with the order of summation reversed.
• We have proved Ramanujan’s Second Identity with the product of the summation indices tending to infinity.
• We have not proved Ramanujan’s Second Identity with the order of summation as written by Ramanujan.
• We have proved Ramanujan’s First Identity under all three interpretations.
• Road Blocks: The plus sign between the two Bessel functions; singularities at 0.
• We have proved Ramanujan’s Second Identity with the order of summation reversed.
• We have proved Ramanujan’s Second Identity with the product of the summation indices tending to infinity.
• We have not proved Ramanujan’s Second Identity with the order of summation as written by Ramanujan.
• We have proved Ramanujan’s First Identity under all three interpretations.
• Road Blocks: The plus sign between the two Bessel functions; singularities at 0.
• The proofs under different interpretations are completely different.
Summary

- We have proved Ramanujan’s Second Identity with the order of summation reversed.
- We have proved Ramanujan’s Second Identity with the product of the summation indices tending to infinity.
- We have not proved Ramanujan’s Second Identity with the order of summation as written by Ramanujan.
- We have proved Ramanujan’s First Identity under all three interpretations.
- Road Blocks: The plus sign between the two Bessel functions; singularities at 0.
- The proofs under different interpretations are completely different.
- New methods of estimating trigonometric sums are introduced.
Riesz Sums

\[ \sum_{n \leq x} a(n)(x - n)^a \]
Analogue of a Theorem of Dixon and Ferrar

**Theorem**

Let $a$ denote a positive integer, and let $\chi$ be an even primitive nonprincipal character of modulus $q$. Set for $x > 0$,

$$D_\chi(a; x) := \frac{1}{\Gamma(a + 1)} \sum_{n \leq x} d_\chi(n)(x - n)^a.$$

Then, for $a \geq 2$,

$$D_\chi(a - 1; x) = \frac{L(1, \chi)x^a}{\Gamma(1 + a)} + \frac{\tau(\chi)q^{a-1}}{(2\pi)^{a-1}} \sum_{n=1}^{\infty} d_\chi(n) \left( \frac{x}{nq} \right)^{a/2}$$

$$\times \left\{ -Y_a \left( 4\pi \sqrt{\frac{nq}{x}} \right) + \frac{2}{\pi} \cos(\pi a)K_a \left( 4\pi \sqrt{\frac{nq}{x}} \right) \right\} + S_a,$$
where $S_a$ is the sum of the residues at the poles $-2m - 1$, $0 \leq m \leq \left[\frac{1}{2}a\right] - 1$, of

$$\frac{\Gamma(s)\zeta(s)L(s, \chi)}{\Gamma(s + a)} \chi^{s+a-1}.$$
The Riesz Sum Generalization of Ramanujan’s Entry 5

**Theorem**

Let $x > 0$, $0 < \theta < 1$, and $a$ be a positive integer. Then,

\[
\frac{1}{(a - 1)!} \sum_{n \leq x}' (x - n)^{a-1} \sum_{r | n} \cos(2\pi r \theta) = \frac{x^{a-1}}{4(a - 1)!} - \frac{x^a}{a!} \log \left( 2 \sin(\pi \theta) \right)
\]

\[
+ \frac{x^{\alpha/2}}{2(2\pi)^{a-1}} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{l_a \left( 4\pi \sqrt{m(n + \theta)} x \right)}{(m(n + \theta))^{a/2}} + \frac{l_a \left( 4\pi \sqrt{m(n + 1 - \theta)} x \right)}{(m(n + 1 - \theta))^{a/2}} \right\}
\]

\[
- \left[ \frac{a}{2} \right] \sum_{k=1}^{\left[ \frac{a}{2} \right]} (-1)^k \zeta(1 - 2k) \left( \zeta(2k, \theta) + \zeta(2k, 1 - \theta) \right) x^{a-2k} \frac{1}{(a - 2k)!(2\pi)^{2k}}, \tag{8}
\]

where $l_{\nu}(x)$ is defined in (3), and $\zeta(s, \alpha)$ denotes the Hurwitz zeta function.
Theorem

Let $I_1(x)$ be defined by (5). If $0 < \theta, \sigma < 1$ and $x > 0$, then

$$
\sum \sum' \left\{ \cos(2\pi n\theta) \cos(2\pi m\sigma) = \frac{1}{4} + \frac{\sqrt{x}}{4} \sum_{n,m \geq 0} \right.
\times \left\{ \begin{array}{c}
I_1(4\pi \sqrt{(n + \theta)(m + \sigma)x}) \\
\sqrt{(n + \theta)(m + \sigma)}
\end{array} \right.
+ \begin{array}{c}
I_1(4\pi \sqrt{(n + 1 - \theta)(m + \sigma)x}) \\
\sqrt{(n + 1 - \theta)(m + \sigma)}
\end{array}
\left. \right\}.
$$
Define, for Dirichlet characters $\chi_1$ modulo $p$ and $\chi_2$ modulo $q$,

$$d_{\chi_1, \chi_2}(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d).$$
Define, for Dirichlet characters $\chi_1$ modulo $p$ and $\chi_2$ modulo $q$,

$$d_{\chi_1,\chi_2}(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d).$$

$$L(2s, \chi_1)L(2s, \chi_2) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^{2s}} \sum_{m=1}^{\infty} \frac{\chi_2(m)}{m^{2s}} = \sum_{n=1}^{\infty} \frac{d_{\chi_1,\chi_2}(n)}{n^{2s}}.$$
An Analogue of Ramanujan’s Entry 5

Define, for Dirichlet characters $\chi_1$ modulo $p$ and $\chi_2$ modulo $q$,

$$d_{\chi_1,\chi_2}(n) = \sum_{d|n} \chi_1(d) \chi_2(n/d).$$

$$L(2s, \chi_1)L(2s, \chi_2) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^{2s}} \sum_{m=1}^{\infty} \frac{\chi_2(m)}{m^{2s}} = \sum_{n=1}^{\infty} \frac{d_{\chi_1,\chi_2}(n)}{n^{2s}}$$

The Functional Equation

$$(\pi^2/(pq))^{-s}\Gamma^2(s)L(2s, \chi_1)L(2s, \chi_2) = \frac{\tau(\chi_1)\tau(\chi_2)}{\sqrt{pq}} \left(\pi^2/(pq)\right)^{-\left(\frac{1}{2} - s\right)}\Gamma^2\left(\frac{1}{2} - s\right)L(1 - 2s, \overline{\chi_1})L(1 - 2s, \overline{\chi_2})$$
Define, for Dirichlet characters $\chi_1$ modulo $p$ and $\chi_2$ modulo $q$,

$$d_{\chi_1, \chi_2}(n) = \sum_{d|n} \chi_1(d) \chi_2(n/d).$$

$$L(2s, \chi_1)L(2s, \chi_2) = \sum_{n=1}^{\infty} \frac{\chi_1(n)}{n^{2s}} \sum_{m=1}^{\infty} \frac{\chi_2(m)}{m^{2s}} = \sum_{n=1}^{\infty} \frac{d_{\chi_1, \chi_2}(n)}{n^{2s}}$$

The Functional Equation

$$(\pi^2/(pq))^{-s} \Gamma^2(s) L(2s, \chi_1)L(2s, \chi_2) = \frac{\tau(\chi_1)\tau(\chi_2)}{\sqrt{pq}} (\pi^2/(pq))^{-(1/2-s)} \Gamma^2(1/2-s) L(1-2s, \overline{\chi_1})L(1-2s, \overline{\chi_2})$$

$$\sum_{n \leq x}^\prime d_{\chi_1, \chi_2}(n) = \frac{\tau(\chi_1)\tau(\chi_2)}{\sqrt{pq}} \sum_{n=1}^{\infty} d_{\chi_1, \chi_2}(n) \left( \frac{x}{n} \right)^{1/2} I_1 \left( 4\pi \sqrt{nx/pq} \right)$$
I had trouble estimating a crucial integral.
I had trouble estimating a crucial integral.

It was suggested that I see Professor Richard Askey,
Hark back to My Ph.D. Thesis

I had trouble estimating a crucial integral.
It was suggested that I see Professor Richard Askey,
who suggested that I see Professor Steve Wainger,
I had trouble estimating a crucial integral.

It was suggested that I see Professor Richard Askey, who suggested that I see Professor Steve Wainger, who suggested that I learn the method of steepest descent, and so I was able to complete my dissertation.
I had trouble estimating a crucial integral.

It was suggested that I see Professor Richard Askey, who suggested that I see Professor Steve Wainger, who suggested that I learn the method of steepest descent, and so I was able to complete my dissertation.

I had trouble estimating a crucial integral.

It was suggested that I see Professor Richard Askey, who suggested that I see Professor Steve Wainger, who suggested that I learn the method of steepest descent, and so I was able to complete my dissertation.


Photograph, Shanghai, July 31, 2013
THE DIRICHLET DIVISOR PROBLEM FOR DICK ASKEY
THE DIRICHLET DIVISOR PROBLEM
FOR DICK ASKEY
Happy 80th Birthday, Dick
THE DIRICHLET DIVISOR PROBLEM FOR DICK ASKEY

Happy 80th Birthday, Dick

and Many more