Askey-Wilson polynomials
and the tetrahedron index

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Work in progress.

Partly joint with Ilmar Gahramanov (Humboldt University of Berlin).
Outline

Encounters with Dick Askey

The tetrahedron index
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Encounters with Dick Askey

The tetrahedron index
Askey 65: Mount Holyoke, 1998
You should look at the $W_9$!
Let $h_k(x; a) = \prod_{j=0}^{k-1} (1 - axq^j + a^2 q^{2j})$ be the Askey-Wilson monomial.

Then, the coefficients in

$$h_k(x; a)h_{N-k}(x; b) = \sum_{l=0}^{N} C_i^k h_l(x; c)h_{N-l}(x; d)$$

are $10W_9$-series.

Some fundamental properties like discrete biorthogonality are immediate consequences.
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Askey 70: Bexbach, 2003

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are $\text{10W}_9$-series.

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This is very simple!
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The best mathematics usually is!
Outline

Encounters with Dick Askey

The tetrahedron index
Recent developments in supersymmetric quantum field theory are interesting from special functions perspective.

“Indices” are typically elliptic hypergeometric integrals for 4D and basic hypergeometric integrals for 3D theories.

“Dual” theories are expected to have the same index \[ \Rightarrow \] Non-trivial integral identities (without rigorous proof).
Integral identities from QFT

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The tetrahedron index was recently introduced by Dimofte, Gaiotto and Gukov. It is

\[ \mathcal{I}_q[m, z] = \frac{(q^{1-m/2} / z; q)_{\infty}}{(q^{-m/2} z; q)_{\infty}} \]

where \( m \in \mathbb{Z}, z \in \mathbb{C}, (a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - aq^j). \)
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where \( m \in \mathbb{Z} \), \( z \in \mathbb{C} \), \((a; q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j)\).

Doesn’t look very exciting yet...
Sample result: “Pentagon identity”

\[
\sum_{m=-\infty}^{\infty} \oint  \frac{(-1)^m}{z^{3m}} \prod_{j=1}^{3} \frac{(q^{1+\frac{m}{2}}/a_jz, q^{1-\frac{m}{2}}z/b_j; q)_{\infty}}{(q^{\frac{m}{2}}a_jz, q^{-\frac{m}{2}}b_j/z; q)_{\infty}} \frac{dz}{2\pi i z} \\
= \prod_{j, k=1}^{3} \frac{(q/a_j b_k; q)_{\infty}}{(a_j b_k; q)_{\infty}},
\]

where \(|q| < 1\) and \(a_1a_2a_3 = b_1b_2b_3 = q^{1/2}\).

Contour can be taken as unit circle if \(|a_j|, |b_j| < 1\).

Derived by Gahramanov using physics arguments and proved by the speaker (Gahramanov & R. 2013).
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Sketch of proof

Shrink contour to zero. Pick up residues at \( z = q^{-\frac{m}{2} + k} b_j \), \( k \geq \max(0, m) \). For \( j = 1 \), the sum of residues is

\[
\text{Const} \cdot \sum_{m=-\infty}^{\infty} \sum_{k=\max(0,m)}^{\infty} \frac{(a_1 b_1, a_2 b_1, a_3 b_1; q)_k}{(q, qb_1/b_2, qb_1/b_3; q)_k} q^k \\
\times \frac{(a_1 b_1, a_2 b_1, a_3 b_1; q)_{k-m}}{(q, qb_1/b_2, qb_1/b_3; q)_{k-m}} q^{k-m} \\
= \text{Const} \cdot \phi_2^2 \left( \frac{a_1 b_1, a_2 b_1, a_3 b_1}{qb_1/b_2, qb_1/b_3; q, q} \right)^2.
\]

We need to prove an identity like

\[
C_1 \cdot \phi_2^2 + C_2 \cdot \phi_2^2 + C_3 \cdot \phi_2^2 = D
\]

(with \( C_j \) and \( D \) explicit infinite products).
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\times \frac{(a_1 b_1, a_2 b_1, a_3 b_1; q)_{k-m}}{(q, qb_1/b_2, qb_1/b_3; q)_{k-m}} q^{k-m}
= \text{Const} \cdot \, _3\phi_2^2 \left( \frac{a_1 b_1, a_2 b_1, a_3 b_1}{qb_1/b_2, qb_1/b_3; q, q} \right)^2 .
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\text{Const} \cdot \sum_{m=-\infty}^{\infty} \sum_{k=\max(0,m)}^{\infty} \frac{(a_1b_1, a_2b_1, a_3b_1; q)_k}{(q, qb_1/b_2, qb_1/b_3; q)_k} q^k \times \frac{(a_1b_1, a_2b_1, a_3b_1; q)_{k-m}}{(q, qb_1/b_2, qb_1/b_3; q)_{k-m}} q^{k-m}
\]

\[
= \text{Const} \cdot \frac{3}{3\phi_2}(a_1b_1, a_2b_1, a_3b_1; q, q)^2.
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C_1 \cdot 3\phi_2^2 + C_2 \cdot 3\phi_2^2 + C_3 \cdot 3\phi_2^2 = D
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Shrink contour to zero. Pick up residues at \( z = q^{-\frac{m}{2}+k}b_j \), \( k \geq \max(0, m) \). For \( j = 1 \), the sum of residues is

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\text{Const} \cdot \sum_{m=-\infty}^{\infty} \sum_{k=\max(0,m)}^{\infty} \frac{(a_1b_1, a_2b_1, a_3b_1; q)_k}{(q, qb_1/b_2, qb_1/b_3; q)_k} q^k
\times \frac{(a_1b_1, a_2b_1, a_3b_1; q)_{k-m}}{(q, qb_1/b_2, qb_1/b_3; q)_{k-m}} q^{k-m}
\]

\[
= \text{Const} \cdot \left( a_1b_1, a_2b_1, a_3b_1; q, q \right)_2^2.
\]

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\[
C_1 \cdot \left( a_1b_1, a_2b_1, a_3b_1; q, q \right)_2^2 + C_2 \cdot \left( a_1b_1, a_2b_1, a_3b_1; q, q \right)_2^2 + C_3 \cdot \left( a_1b_1, a_2b_1, a_3b_1; q, q \right)_2^2 = D
\]

(with \( C_j \) and \( D \) explicit infinite products).
Continued proof

Start from

\[(x_2 - x_1)(x_3 - x_1)(x_3 - x_2) = \begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix}\]

\[= (x_3 - x_2)x_1^2 + (x_1 - x_3)x_2^2 + (x_2 - x_1)x_3^2.\]

Substitute

\[x_1 = \frac{b_1(qb_1/b_2, qb_1/b_3; q)_\infty}{(a_1b_1, a_2b_1, a_3b_1; q)_\infty} 3\phi_2 \left( \begin{array}{c} a_1b_1, a_2b_1, a_3b_1 \\ qb_1/b_2, qb_1/b_3 \end{array} ; q, q \right),\]

\[x_2\] and \[x_3\] obtained by permuting \(b_j\).

Then \(x_i - x_j\) factors by non-terminating \(q\)-Saalschütz summation.
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Substitute

\[x_1 = \frac{b_1(qb_1/b_2, qb_1/b_3; q)_\infty}{(a_1b_1, a_2b_1, a_3b_1; q)_\infty} \, {}_3\phi_2 \left( \begin{array}{c} a_1b_1, a_2b_1, a_3b_1 \\ qb_1/b_2, qb_1/b_3 \end{array} ; q, q \right), \]

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\end{vmatrix} = (x_3 - x_2)x_1^2 + (x_1 - x_3)x_2^2 + (x_2 - x_1)x_3^2.\]

Substitute

\[x_1 = \frac{b_1(q b_1 / b_2, q b_1 / b_3; q)_\infty}{(a_1 b_1, a_2 b_1, a_3 b_1; q)_\infty} \, _3\phi_2 \left( \begin{array}{c}
a_1 b_1, a_2 b_1, a_3 b_1 \\
qb_1 / b_2, q b_1 / b_3 \\
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\[x_2 \text{ and } x_3 \text{ obtained by permuting } b_j.\]
Then \(x_i - x_j\) factors by non-terminating \(q\)-Saalschütz summation.
Extended pentagon identity

\[
\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{3} \frac{(q^{1+m/2}/a_j z, q^{1-m/2}z/b_j; q)_\infty}{(q^{M_j+m/2}a_j z, q^{N_j-m/2}b_j/z; q)_\infty} \frac{dz}{2\pi i z} = \prod_{j, k=1}^{3} \frac{(q/a_j b_k; q)_\infty}{(a_j b_k q^{M_j+N_k}; q)_\infty},
\]

where \(a_1 a_2 a_3 = b_1 b_2 b_3 = q^{1/2}\), \(M_j\) and \(N_j\) are integers with \(M_1 + M_2 + M_3 = N_1 + N_2 + N_3 = 0\).

Found by speaker by extending proof above, then Gahramanov found a physics derivation.
Extended pentagon identity

\[ \sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{3} \frac{(q^{1+m/2}/a_j z, q^{1-m/2}/b_j; q)_\infty}{(q^{M_j+m/2}a_j z, q^{N_j-m/2}b_j/z; q)_\infty} \frac{dz}{2\pi i z} \]

\[ = \prod_{j, k=1}^{3} \frac{(q/a_j b_k; q)_\infty}{(a_j b_k q^{M_j+N_k}; q)_\infty}, \]

where \( a_1 a_2 a_3 = b_1 b_2 b_3 = q^{1/2} \), \( M_j \) and \( N_j \) are integers with \( M_1 + M_2 + M_3 = N_1 + N_2 + N_3 = 0. \)

Found by speaker by extending proof above, then Gahramanov found a physics derivation.
GR: Identity discussed above.
Arrows are only formal limits.
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Nassrallah–Rahman type

\[ \sum_{m=-\infty}^{\infty} \int \prod_{j=1}^{6} \frac{(q^{1+m/2}/a_j z, q^{1-m/2} z/a_j; q)_{\infty}}{(q^{N_j+m/2} a_j z, q^{N_j-m/2} a_j z; q)_{\infty}} \]

\[ \times \frac{(1 - q^m z^2)(1 - q^m z^{-2})}{q^m z^{6m}} \frac{dz}{2\pi i z} \]

\[ = \frac{2}{\prod_{j=1}^{6} q^{\binom{N_j}{2}} a_j^{N_j}} \prod_{1 \leq j < k \leq 6} \frac{(q/a_j a_k; q)_{\infty}}{(a_j a_k q^{N_j+N_k}; q)_{\infty}}, \]

\[ a_1 \cdots a_6 = q \text{ and } N_1 + \cdots + N_6 = 0. \]
Askey–Wilson-type

\[
\sum_{m=-\infty}^{\infty} \int \prod_{j=1}^{4} \frac{(q^{1+m/2}/a_j z, q^{1-m/2} z/a_j; q)_{\infty}}{(q^{m/2} b_j z, q^{-m/2} b_j /z; q)_{\infty}} \\
\times \frac{(1 - q^m z^2)(1 - q^m z^{-2})}{q^m z^{4m}} \frac{dz}{2\pi i z} = \frac{2(b_1 b_2 b_3 b_4; q)_{\infty}}{(q/a_1 a_2 a_3 a_4; q)_{\infty}} \prod_{1 \leq j < k \leq 4} \frac{(q/a_j a_k; q)_{\infty}}{(b_j b_k; q)_{\infty}}.
\]

Need \(|q/a_1 a_2 a_3 a_4| < 1\) for convergence.

Compare Askey–Wilson integral

\[
\int \frac{(z^2, z^{-2}; q)_{\infty}}{\prod_{j=1}^{4} (b_j z, b_j /z; q)_{\infty}} \frac{dz}{2\pi i z} = \frac{2(b_1 b_2 b_3 b_4; q)_{\infty}}{\prod_{1 \leq j < k \leq 4} (b_j b_k; q)_{\infty}}.
\]
Askey–Wilson-type

\[
\sum_{m=-\infty}^{\infty} \int_0^4 \prod_{j=1}^{4} \frac{(q^{1+m/2}/a_jz, q^{1-m/2}z/a_j; q)_\infty}{(q^{m/2}b_jz, q^{-m/2}b_j/z; q)_\infty} \\
\times \frac{(1 - q^mz^2)(1 - q^mz^{-2})}{q^mz^{4m}} \frac{dz}{2\pi iz} \\
= \frac{2(b_1b_2b_3b_4; q)_\infty}{(q/a_1a_2a_3a_4; q)_\infty} \prod_{1\leq j<k\leq 4} \frac{(q/a_ja_k; q)_\infty}{(b_jb_k; q)_\infty}.
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Need \(|q/a_1a_2a_3a_4| < 1\) for convergence.

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\int_0^4 \frac{(z^2, z^{-2}; q)_\infty}{\prod_{j=1}^{4} (b_jz, b_j/z; q)_\infty} \frac{dz}{2\pi iz} = \frac{2(b_1b_2b_3b_4; q)_\infty}{\prod_{1\leq j<k\leq 4} (b_jb_k; q)_\infty}.
\]
Proof of Askey–Wilson-type identity

Much easier than what we did before!
Poles at $z = q^{k-m/2}/b_j$, $k \geq 0$. Shift $z \mapsto q^{-m/2}z$.

$$\sum_{m=-\infty}^{\infty} \oint \prod_{j=1}^{4} \frac{(q^{1+m}/a_jz, q^{1-m}z/a_j; q)_{\infty}}{(b_jz, b_j/z; q)_{\infty}} \times \frac{(1 - z^2)(1 - q^{2m}z^{-2})}{q^mz^{4m}} \frac{dz}{2\pi i z}$$

Interchange sum and integral. Sum is summable $\psi_6$. Integral is Askey–Wilson integral.
Proof of Askey–Wilson-type identity

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Poles at $z = q^{k-m/2}/b_j$, $k \geq 0$. Shift $z \mapsto q^{-m/2}z$.

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\sum_{m=-\infty}^{\infty} \int_0^4 \prod_{j=1}^4 \frac{(q^{1+m}/a_j z, q^{1-m}z/a_j; q)_\infty}{(b_j z, b_j/z; q)_\infty}
\times \frac{(1 - z^2)(1 - q^{2m}z^{-2})}{q^m z^{4m}} \frac{dz}{2\pi i z}
$$

Interchange sum and integral. Sum is summable $6\psi_6$. Integral is Askey–Wilson integral.
What can we integrate?

Replacing $a_1 \mapsto a_1 q^k$, $b_1 \mapsto b_1 q^l$ gives
(recall that $h_k(x + x^{-1}; a) = (ax, a/x; q)_k$)

$$\sum_{m=-\infty}^{\infty} \int \prod_{j=1}^{4} \frac{(q^{1+m/2}/az, q^{1-m/2}z/aj; q)_\infty}{(q^{m/2}bz, q^{-m/2}bj/z; q)_\infty} \prod_{j=1}^{4} \frac{(q^m z^2)(1 - q^m z^2)}{q^m z^{4m}} dz$$

$$= \frac{2(b_1 b_2 b_3 b_4; q)_\infty \prod_{1 \leq j < k \leq 4} (q/ajak; q)_\infty}{(q/a_1 a_2 a_3 a_4; q)_\infty} \prod_{j=2}^{4} (a_1 a_j; q)_k (b_1 b_j; q)_l$$

$$= (a_1 a_2 a_3 a_4; q)_k (b_1 b_2 b_3 b_4; q)_l.$$
Decoupling phenomenon

On the other hand,

\[
\prod_{j=2}^{4} (a_1 a_j; q)_k (b_1 b_j; q)_l \over (a_1 a_2 a_3 a_4; q)_k (b_1 b_2 b_3 b_4; q)_l
\]

\[
= \text{Const} \cdot \int \prod_{j=1}^{4} (a_j z, a_j / z; q)_\infty \frac{h_k(z + z^{-1}; a_1)}{2\pi iz} \frac{dz}{2\pi iz} \times \prod_{j=1}^{4} (b_j z, b_j / z; q)_\infty \frac{h_l(z + z^{-1}; b_1)}{2\pi iz} \frac{dz}{2\pi iz}.
\]

Our \( \sum \oint \) “decouples” as product of two Askey–Wilson integrals.
Askey-Wilson polynomials

Polynomials orthogonal with respect to Askey–Wilson integral, denoted
\[ p_n\left(\frac{z + z^{-1}}{2}; a_1, a_2, a_3, a_4; q\right). \]

By decoupling phenomenon, the polynomials
\[
p_k\left(\frac{zq^{-m/2} + z^{-1}q^{m/2}}{2}; a_1, a_2, a_3, a_4; q\right)
\]
\[
p_l\left(\frac{zq^{m/2} + z^{-1}q^{-m/2}}{2}; b_1, b_2, b_3, b_4; q\right)
\]
are orthogonal with respect to our sum. 

Caveat: For convergence, we need \( |q^{1-k}/a_1a_2a_3a_4| < 1 \), so \( k \) runs only over a finite set.
Askey-Wilson polynomials

Polynomials orthogonal with respect to Askey–Wilson integral, denoted
\[ p_n((z + z^{-1})/2; a_1, a_2, a_3, a_4; q). \]

By decoupling phenomenon, the polynomials

\[ p_k \left( \frac{z q^{-m/2} + z^{-1} q^{m/2}}{2}; a_1, a_2, a_3, a_4; q \right) \]

\[ p_l \left( \frac{z q^{m/2} + z^{-1} q^{-m/2}}{2}; b_1, b_2, b_3, b_4; q \right) \]

are orthogonal with respect to our \( \sum \int \).

Caveat: For convergence, we need \(|q^{1-k}/a_1 a_2 a_3 a_4| < 1\), so \( k \) runs only over a finite set.
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By decoupling phenomenon, the polynomials

\[
p_k \left( \frac{z q^{-m/2} + z^{-1} q^{m/2}}{2}; a_1, a_2, a_3, a_4; q \right)
\]

\[
p_l \left( \frac{z q^{m/2} + z^{-1} q^{-m/2}}{2}; b_1, b_2, b_3, b_4; q \right)
\]

are orthogonal with respect to our \( \sum \oint \).

Caveat: For convergence, we need \( |q^{1-k}/a_1 a_2 a_3 a_4| < 1 \), so \( k \) runs only over a finite set.
What am I forgetting?
What am I forgetting?

You should look at the $\mathcal{W}_9$!
What about the $10W_9$?

For the top level “Nasrallah-Rahman”-type $\sum \phi$, there is no convergence problem.

Might lead to new biorthogonality relation for products of two $10W_9$-series, but I haven’t worked out the details yet.
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Final slide for today

Happy birthday Dick!