Ramanujan's theories of elliptic functions to alternative bases, and beyond.

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Outline

Roger Apéry, 1916–1994

Classical results

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}, \ldots$$

Apéry's theorem (1978)

$$\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3} \quad \text{is irrational}$$

R. Apéry

Apéry introduced two sequences of integers: one to prove $\zeta(2) \notin \mathbb{Q}$ (well-known) and the other to prove $\zeta(3) \notin \mathbb{Q}$ (new).

Apéry's numbers used to prove $\zeta(2) \notin \mathbb{Q}$

$$(k+1)^2 s_{k+1} = (11k^2 + 11k + 3)s_k + k^2 s_{k-1}, \quad s_0 = 1$$

The numbers s_k are integers—not obvious.

Apéry's numbers used to prove $\zeta(3) \notin \mathbb{Q}$

$$(k+1)^3 t_{k+1} = (2k+1)(17k^2 + 17k + 5)t_k - k^3 t_{k-1}, \quad t_0 = 1$$

The numbers t_k are integers—not obvious.

Apéry's numbers used to prove $\zeta(2) \notin \mathbb{Q}$

$$(k+1)^2 s_{k+1} = (11k^2 + 11k + 3)s_k + k^2 s_{k-1}, \quad s_0 = 1$$

$$s_k = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{k+j}{j}$$

Proof: by the method of creative telescoping. Now, automated.

```
> with(sumtools):

> sumrecursion(binomial(k,j)^2*binomial(k+j,j), j, s(k));
-(k-1)^2 s(-2+k) - (11 k^2 - 11 k + 3) s(k-1)
    + s(k) k^2
```

The numbers s_k are integers—now obvious from the binomial sum.

F. Beukers, 1983

$$(k+1)^2 s_{k+1} = (11k^2 + 11k + 3)s_k + k^2 s_{k-1}, \quad s_0 = 1$$

$$z = \sum_{k=0}^{\infty} s_k x^k$$

$$x = q \prod_{j=1}^{\infty} \frac{(1-q^{5j-4})^5 (1-q^{5j-1})^5}{(1-q^{5j-3})^5 (1-q^{5j-2})^5} = r_5(q)^5$$

$$z = \prod_{j=1}^{\infty} \frac{(1-q^j)^2}{(1-q^{5j-4})^5 (1-q^{5j-1})^5}$$

1. Another proof that s_k is an integer
2. $r_5(q)$: the Rogers-Ramanujan continued fraction
3. How to find other examples?

Sporadic sequences

Franel, 1894

$$(k+1)^2 s_{k+1} = (7k^2+7k+2)s_k + 8k^2 s_{k-1}, \quad s_0 = 1$$

$$s_k = \sum_{j=0}^{k} \binom{k}{j}^3$$

Apéry, 1978

$$(k+1)^2 s_{k+1} = (11k^2+11k+3)s_k + k^2 s_{k-1}, \quad s_0 = 1$$

$$s_k = \sum_{j=0}^{k} \binom{k}{j}^2 \binom{k+j}{j}$$

Zagier, 1998, 2009

$$(k+1)^2 s_{k+1} = (ak^2+ak+b)s_k + ck^2 s_{k-1}, \quad s_0 = 1$$

$(k+1)^2 s_{k+1} = (ak^2 + ak + b)s_k + ck^2 s_{k-1}, \quad s_0 = 1$

(a, b, c)	$s(k)$
$(11, 3, 1)$	$\sum_j \binom{k}{j}^2 \binom{k+j}{j}$
$(-17, -6, -72)$	$\sum_{j,\ell} (-8)^{k-j} \binom{k}{j} \binom{j}{\ell}^3$
$(10, 3, -9)$	$\sum_j \binom{k}{j}^2 \binom{2j}{j}$
$(7, 2, 8)$	$\sum_j \binom{k}{j}^3$
$(12, 4, -32)$	$\sum_j 4^{k-2j} \binom{k}{2j} \binom{2j}{j}^2$
$(-9, -3, -27)$	$\sum_j (-3)^{k-3j} \binom{k}{j} \binom{k-j}{j} \binom{k-2j}{j}$

Analogues of Beukers' result

$$(k+1)^2 s_{k+1} = (7k^2+7k+2)s_k + 8k^2 s_{k-1}, \quad s_0 = 1$$

$$z = \sum_{k=0}^{\infty} s_k x^k = \sum_{k=0}^{\infty} \left\{ \sum_{j=0}^{k} \binom{k}{j}^3 \right\} x^k$$

$$x = q \prod_{j=1}^{\infty} \frac{(1-q^j)^3(1-q^{6j})^9}{(1-q^{2j})^3(1-q^{3j})^9} = r_c(q)^3$$

$$z = \prod_{j=1}^{\infty} \frac{(1-q^{2j})(1-q^{3j})^6}{(1-q^j)^2(1-q^{6j})^3}$$

1. $r_c(q)$ is Ramanujan's cubic continued fraction
2. Similar results hold for Zagier's other examples

Apéry's numbers used to prove $\zeta(2) \notin \mathbb{Q}$

$$(k+1)^2 s_{k+1} = (11k^2 + 11k + 3)s_k + k^2 s_{k-1}, \quad s_0 = 1$$

1. Three-term recurrence relation
2. Coefficients are polynomials of degree 2
3. Zagier's sporadic sequences

$$(k+1)^2 s_{k+1} = (ak^2 + ak + b)s_k + ck^2 s_{k-1}, \quad s_0 = 1$$

Apéry's numbers used to prove $\zeta(3) \notin \mathbb{Q}$

$$(k+1)^3 t_{k+1} = (2k+1)(17k^2 + 17k + 5)t_k - k^3 t_{k-1}, \quad t_0 = 1$$

1. Three-term recurrence relation
2. Coefficients are polynomials of degree 3

$$(k+1)^3 t_{k+1} = (2k+1)(17k^2+17k+5)t_k - k^3 t_{k-1}, \quad t_0 = 1$$

$$x = q \prod_{j=1}^{\infty} \frac{(1-q^j)^{12}(1-q^{6j})^{12}}{(1-q^{2j})^{12}(1-q^{3j})^{12}}$$

$$(k+1)^3 t_{k+1} = -(2k+1)(11k^2+11k+5)t_k - 125k^3 t_{k-1}, \quad t_0 = 1$$

Analogues of

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{4k}{2k} \frac{(1103+26390k)}{396^{4k}}.$$

H. H. Chan and coauthors, series of papers

Apéry numbers and modular forms

$$\begin{aligned} f(5) &:= \frac{5P(q^5) - P(q)}{4} \\ &= \sum_{j=0}^{\infty} \binom{2j}{j} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k} \right\} \left(\frac{\eta_1^2 \eta_5^2}{f(5)} \right)^{2j} \end{aligned}$$

$$\begin{aligned} f(6) &:= \frac{30P(q^6) - 3P(q^3) + 2P(q^2) - 5P(q)}{24} \\ &= \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k}^2 \right\} \left(\frac{\eta_1 \eta_2 \eta_3 \eta_6}{f(6)} \right)^{2j} \end{aligned}$$

$$\eta_n = q^{n/24} \prod_{j=1}^{\infty} (1 - q^{nj}), \qquad P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1 - q^j}$$

Outline

Jacobi, 1829

$$x = \left(\frac{\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \right)^4, \qquad |q| < 1$$

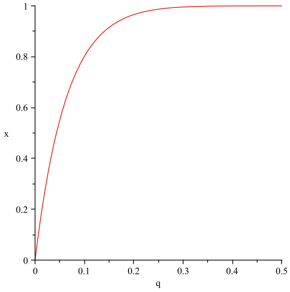


Figure: Graph of x versus q

$$\begin{aligned} x(0) &= 0 \\ x(1) &= 1 \\ \frac{dx}{dq} &> 0 \\ \left.\frac{dx}{dq}\right|_{q=0} &= 16 \end{aligned}$$

Hypergeometric function

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad n \in \mathbf{Z}^+; \quad (a)_0 = 1$$

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n n!} x^n$$

$${}_3F_2(a,b,c;d,e;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n n!} x^n$$

$${}_0F_0(-;-;x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$${}_1F_0(a;-;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n$$

Jacobi's inversion formula, 1829

$$x = \left(\frac{\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \right)^4, \qquad q = \exp\left(-\pi \, \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)} \right)$$

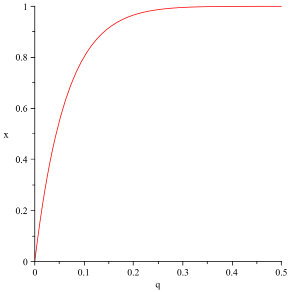


Figure: Graph of x versus q

$$\begin{aligned} x(0) &= 0 \\ x(1) &= 1 \\ x(e^{-\pi}) &= 1/2 \\ x(e^{-\pi t}) + x(e^{-\pi/t}) &= 1, \quad t > 0 \end{aligned}$$

Jacobi's inversion formula, 1829

$$x = \left(\frac{\sum_{n=-\infty}^{\infty} q^{(n+\frac{1}{2})^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}} \right)^4, \qquad q = \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)} \right)$$

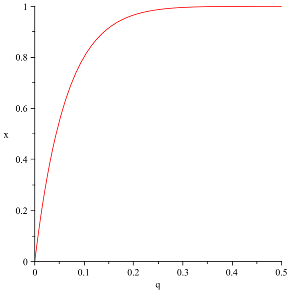


Figure: Graph of x versus q

$$z = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x) = \left(\sum_{n=-\infty}^{\infty} q^{n^2} \right)^2$$

$$q \frac{dx}{dq} = z^2 x(1-x)$$

Squares of hypergeometric functions

$${}_0F_0\left(-;-;x\right)^2 = \left(e^x\right)^2 = {}_0F_0\left(-;-;2x\right)$$

$${}_1F_0\left(a;-;x\right)^2 = \left(\left(1-x\right)^{-a}\right)^2 = {}_1F_0\left(2a;-;x\right)$$

$${}_2F_1\left(a,b;c;x\right)^2 =$$

Clausen's identity (1828)

Clausen's identity

$$
{}_2F_1\left(\frac{1}{2}+\epsilon,\frac{1}{2}-\epsilon;1;x\right)^2 = {}_3F_2\left(\frac{1}{2}+\epsilon,\frac{1}{2}-\epsilon,\frac{1}{2};1,1;4x(1-x)\right)
$$

Clausen + Jacobi

$$
\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 = {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;4x(1-x)\right)
$$

After some manipulations (this expression generalizes)

$$
f(4) := \frac{4P(q^4)-P(q)}{3} = \sum_{j=0}^{\infty}\binom{2j}{j}^3\left(\frac{\eta_1^4\eta_4^4}{\eta_2^4\, f(4)}\right)^{2j}
$$

Aside: Jacobi's sum of four squares theorem

$$\left(\sum_{n=-\infty}^{\infty} q^{n^2}\right)^4 = 1 + 8 \sum_{\substack{j=1 \\ 4\nmid j}}^{\infty} \frac{jq^j}{1-q^j}$$

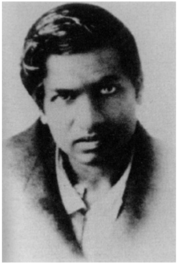
$$\#\left\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = n,\ x_1, x_2, x_3, x_4 \in \mathbb{Z}\right\} = 8 \sum_{\substack{d \mid n \\ 4\nmid d}} d$$

Corollary (Lagrange) Every positive integer is a sum of four squares.

$$\#\left\{x_1^2 + x_2^2 + x_3^2 + x_4^2 = n\right\} > 0$$

Another aside: Ramanujan (1914)

$$\frac{1}{\pi} = \frac{1}{16} \sum_{j=0}^{\infty} \binom{2j}{j}^3 \frac{(42j+5)}{2^{12j}}$$



Outline

Ramanujan (1914)

There are similar theories when

$$q = \exp\left(-\pi\,\frac{z(1-x)}{z(x)}\right), \qquad z(x) = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x\right)$$

is replaced by any of

$$q_1 = \exp\left(-\pi\sqrt{2}\,\frac{z_1(1-x)}{z_1(x)}\right), \qquad z_1(x) = {}_2F_1\left(\frac{1}{4},\frac{3}{4};1;x\right)$$

$$q_2 = \exp\left(-\frac{2\pi}{\sqrt{3}}\,\frac{z_2(1-x)}{z_2(x)}\right), \qquad z_2(x) = {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right)$$

$$q_3 = \exp\left(-2\pi\,\frac{z_3(1-x)}{z_3(x)}\right), \qquad z_3(x) = {}_2F_1\left(\frac{1}{6},\frac{5}{6};1;x\right)$$

Ramanujan's "alternative theories" of elliptic functions

Ramanujan (1914)

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}}\,\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right)}\right)$$

J. M. Borwein and P. B. Borwein (1991)

$$x = \left(\frac{\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{(m+\frac{1}{3})^2+(m+\frac{1}{3})(n+\frac{1}{3})+(n+\frac{1}{3})^2}}{\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2}}\right)^3$$

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}} \frac{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; 1-x\right)}{{}_2F_1\left(\frac{1}{3}, \frac{2}{3}; 1; x\right)} \right)$$

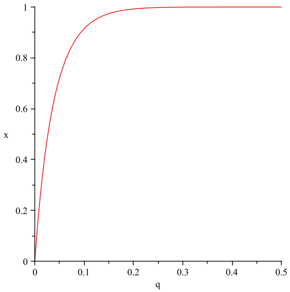


Figure: Graph of x versus q

$$\begin{aligned} x(0) &= 0 \\ x(1) &= 1 \\ x(e^{-2\pi/\sqrt{3}}) &= 1/2 \end{aligned}$$

Ramanujan's "alternative theories" of elliptic functions

$$q = \exp\left(-\frac{2\pi}{\sqrt{3}}\ \frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-x\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right)}\right)$$

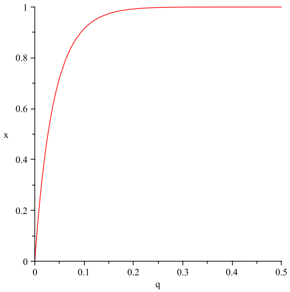


Figure: Graph of x versus q

$$\begin{aligned} z &= {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;x\right) \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} q^{m^2+mn+n^2} \\ q\frac{dx}{dq} &= z^2x(1-x) \end{aligned}$$

Ramanujan's "alternative theories" of elliptic functions

Ramanujan

- pp. 257–262, second notebook
- 27 Feb 1913, second letter to G. H. Hardy
- 1914 paper "Modular equations and approximations to $1/\pi$"
  17 series for $1/\pi$

Mordell (1927), Watson (1931)

- "It is unfortunate that Ramanujan has not developed in detail the corresponding theories..."
- "There are developments of functions analogous to elliptic functions which I have not seen elsewhere..."

Fricke (1916)

Inversion formula for ${}_2F_1(\frac{1}{6}, \frac{5}{6}; 1; x)$

Ramanujan's "alternative theories" of elliptic functions

K. Venkatachaliengar (1988, republished 2012)

Initial investigations into the "alternative theories"

J. M. Borwein and P. B. Borwein (1987–1994)

- A book and a series of papers
- Proved all 17 of Ramanujan's series for $1/\pi$
- Discovered the cubic theta function $\sum\sum q^{m^2+mn+n^2}$

Berndt, Bhargava and Garvan (1995)

Proved all of the results on pp. 257–262 of Ramanujan's second notebook. (Trans. Amer. Math. Soc., 82 pages)

Ramanujan's "alternative theories" of elliptic functions

H. H. Chan (1998)

The ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; x)$ theory

Berndt, Chan and Liaw (2001)

The ${}_2F_1(\frac{1}{4}, \frac{3}{4}; 1; x)$ theory

K. S. Williams (2004)

The ${}_2F_1(\frac{1}{3}, \frac{2}{3}; 1; x)$ theory

C., (2009)

A unified treatment for all four theories

D. Schultz (2013)

Cubic theory

Example of a modular form

Eisenstein series

$$E_{2n}(\tau) = \sum_{(j,k)\neq(0,0)} \frac{1}{(j+k\tau)^{2n}}, \quad n = 2, 3, \ldots, \quad \mathrm{Im}(\tau) > 0$$

Transformations

$$\begin{aligned} E_{2n}(\tau+1) &= E_{2n}(\tau) \\ E_{2n}\left(-\frac{1}{\tau}\right) &= \tau^{2n} E_{2n}(\tau) \end{aligned}$$

E_{2n} is a modular form of weight $2n$

$$E_{2n}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^{2n} E_{2n}(\tau)$$

for all integers a, b, c and d with $ad - bc = 1$.

The effect of scaling

Suppose

$$f\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^n f(\tau)$$

for all integers a, b, c and d with $ad-bc=1$.

Let m be a positive integer and let $g(\tau)=f(m\tau)$. Then

$$g\left(\frac{a\tau+b}{c\tau+d}\right)=(c\tau+d)^n g(\tau)$$

for all integers a, b, c and d with $ad-bc=1$, provided in addition $c \equiv 0 \pmod{m}$.

Congruence subgroups

The modular group

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbf{Z}, \; ad - bc = 1 \right\}$$

Congruence subgroup

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \middle| \, c \equiv 0 \pmod{m} \right\}$$

Modular form

A function f is a modular form of weight n and level m if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^n f(\tau) \quad \text{for all} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(m).$$

Examples of modular forms

Suppose $q = \exp(2\pi i\tau)$ so $\operatorname{Im}(\tau) > 0 \iff |q| < 1$

Let $P(\tau) = 1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}$ $\quad Q(\tau) = 1 + 240\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}$

- $P\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2P(\tau) - \pi ic(c\tau+d)$.
- $P(\tau)$ is not a modular form
- $mP(m\tau) - P(\tau)$ is a modular form of weight 2 and level m

- $Q(\tau)$ is a modular form of weight 4 (and level 1)

Ramanujan's alternative theories of elliptic functions

$$z_1^4 = Q(\tau) \qquad z_1 = {}_2F_1\left(\tfrac{1}{6},\tfrac{5}{6};1;x_1\right)$$

$$z_2^2 = 2P(2\tau) - P(\tau) \qquad z_2 = {}_2F_1\left(\tfrac{1}{4},\tfrac{3}{4};1;x_2\right)$$

$$z_3^2 = \frac{3P(3\tau) - P(\tau)}{2} \qquad z_3 = {}_2F_1\left(\tfrac{1}{3},\tfrac{2}{3};1;x_3\right)$$

$$z_4^2 = \frac{4P(4\tau) - P(\tau)}{3} \qquad z_4 = {}_2F_1\left(\tfrac{1}{2},\tfrac{1}{2};1;x_4\right)$$

$$q\frac{dx_m}{dq} = z_m^2 x_m(1 - x_m), \qquad x_m(e^{-2\pi/\sqrt{m}}) = \frac{1}{2}$$

A common way of viewing all 4 theories

$$f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left(\frac{\eta_1^4 \eta_4^4}{\eta_2^4\ f(4)} \right)^{2j}$$

$$f(3) := \frac{3P(q^3) - P(q)}{2} = \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{3j}{j} \left(\frac{\eta_1^2 \eta_3^2}{f(3)} \right)^{3j}$$

$$f(2) := 2P(q^2) - P(q) = \sum_{j=0}^{\infty} \binom{2j}{j}^2 \binom{4j}{2j} \left(\frac{\eta_1^2 \eta_2^2}{f(2)} \right)^{4j}$$

$$f(1) := Q(q)^{1/2} = \sum_{j=0}^{\infty} \binom{2j}{j} \binom{3j}{j} \binom{6j}{3j} \left(\frac{\eta_1^4}{f(1)} \right)^{6j}$$

$$P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1 - q^j}, \qquad Q(q) = 1 + 240 \sum_{j=1}^{\infty} \frac{j^3 q^j}{1 - q^j}$$

$$\eta_m = q^{m/24} \prod_{j=1}^{\infty} (1 - q^{mj})$$

Outline

F. Beukers, 1983

$$(k+1)^2 s_{k+1} = (11k^2 + 11k + 3)s_k + k^2 s_{k-1}, \quad s_0 = 1$$

$$z = \sum_{k=0}^{\infty} s_k x^k$$

$$x = q \prod_{j=1}^{\infty} \frac{(1-q^{5j-4})^5 (1-q^{5j-1})^5}{(1-q^{5j-3})^5 (1-q^{5j-2})^5} = r_5(q)^5$$

$$z = \prod_{j=1}^{\infty} \frac{(1-q^j)^2}{(1-q^{5j-4})^5 (1-q^{5j-1})^5}$$

How does it fit in with Jacobi and Ramanujan's theories?

Analogues of Clausen's formula

Chan, Tanigawa, Yang and Zudilin (2011): Clausen 1

$$(1+cw^2)\left(\sum_{j=0}^{\infty} u_j w^j\right)^2 = \sum_{j=0}^{\infty} \binom{2j}{j} u_j \left(\frac{w(1-aw-cw^2)}{(1+cw^2)^2}\right)^j$$

Almkvist, van Straten and Zudilin (2011): Clausen 2

$$(1-aw-cw^2)\left(\sum_{j=0}^{\infty} u_j w^j\right)^2 = \sum_{j=0}^{\infty} t_j \left(\frac{w}{1-aw-cw^2}\right)^j$$

$$(j+1)^2 u_{j+1} = (aj^2+aj+b)u_j + cj^2 u_{j-1}$$

$$(j+1)^3 t_{j+1} = -(2j+1)(aj^2+aj+a-2b)t_j - (4c+a^2)j^3 t_{j-1}$$

$$u_0 = t_0 = 1$$

Higher levels: Clausen 1 example

$$f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left(\frac{\eta_1^4 \eta_4^4}{\eta_2^4\ f(4)} \right)^{2j}$$

$$f(5) := \frac{5P(q^5) - P(q)}{4}$$
$$= \sum_{j=0}^{\infty} \binom{2j}{j} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k} \right\} \left(\frac{\eta_1^2 \eta_5^2}{f(5)} \right)^{2j}$$

$$s_j = \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k}$$

$$(j+1)^2 s_{j+1} = (11j^2 + 11j + 3)s_j + j^2 s_{j-1}$$

R. Apéry: $\zeta(2) \notin \mathbb{Q}$

Rogers-Ramanujan continued fraction

$$\frac{5P(q^5)-P(q)}{4} = \sum_{j=0}^{\infty} \binom{2j}{j} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k} \right\} \left(\frac{\eta_1^2 \eta_5^2}{f(5)} \right)^{2j}$$

$$r = r(q) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}.$$

$$\left(\frac{\eta_1^2 \eta_5^2}{f(5)} \right)^2 = \frac{r^5(1 - 11r^5 - r^{10})}{(1 + r^{10})^2}.$$

Higher levels: Clausen 2 example

$$f(4) := \frac{4P(q^4) - P(q)}{3} = \sum_{j=0}^{\infty} \binom{2j}{j}^3 \left(\frac{\eta_1^4 \eta_4^4}{\eta_2^4\, f(4)} \right)^{2j}$$

$$f(6) := \frac{30P(q^6) - 3P(q^3) + 2P(q^2) - 5P(q)}{24}$$

$$= \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{j} \binom{j}{k}^2 \binom{j+k}{k}^2 \right\} \left(\frac{\eta_1 \eta_2 \eta_3 \eta_6}{f(6)} \right)^{2j}$$

$(j+1)^3 t_{j+1} = (2j+1)(17j^2+17j+5)t_j - j^3 t_{j-1}$: Apéry, $\zeta(3) \notin \mathbb{Q}$

$$P(q) = 1 - 24 \sum_{j=1}^{\infty} \frac{jq^j}{1-q^j}, \qquad \eta_m = q^{m/24} \prod_{j=1}^{\infty} (1 - q^{mj})$$

Summary, so far

Levels 1, 2, 3: Ramanujan's theories to alternative bases

Level 4: Jacobi

Levels 5, 6, 6, 6, 8, 9: Zagier's sporadic sequences

${}_2F_1$ functions correspond to weight one modular forms.
(e.g., elliptic integral $\leftrightarrow$ sum of two squares)

$$(j+1)^2 u_{j+1} = (aj^2 + aj + b)u_j + cj^2 u_{j-1}$$

${}_3F_2$ functions correspond to weight two modular forms.

$$(j+1)^3 t_{j+1} = -(2j+1)(aj^2 + aj + a - 2b)t_j - (4c + a^2)j^3 t_{j-1}$$

$$(j+1)^3 s_{j+1} = 2(2j+1)(aj^2 + aj + b)s_j + 4cj(4j^2 - 1)s_{j-1}$$

Other three-term recurrence relations

$$s_j = \sum_{k=0}^{j} \binom{j}{k}^4, \qquad \text{level } 10$$

$$(j+1)^3 s_{j+1} = (2j+1)(6j^2+6j+2)s_j + j(64j^2-4)s_{j-1}, \quad \text{Franel}$$

Experimental search:

$$(j+1)^3 s_{j+1} = (2j+1)(aj^2+aj+b)s_j + j(cj^2+d)s_{j-1}$$

(a,b,c,d)	$(13,4,27,-3)$	$(6,2,64,-4)$	$(14,6,-192,12)$
level	7	10	18

Level 7

$$f(7) = \frac{7P(q^7) - P(q)}{6} = \left(\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{j^2+jk+2k^2} \right)^2$$

$$= \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{\lfloor j/2 \rfloor} \binom{j}{k}^2 \binom{2j-k}{j} \binom{2j-2k}{j} \right\} \left(\frac{\eta_1^2 \eta_7^2}{f(7)} \right)^{3j/2}.$$

Level 10

$$f(10) := \frac{10P(q^{10}) + 5P(q^5) - 2P(q^2) - P(q)}{12}$$

$$= \sum_{j=0}^{\infty} \left\{ \sum_{k=0}^{j} \binom{j}{k}^4 \right\} \left(\frac{\eta_1 \eta_2 \eta_5 \eta_{10}}{f(10)} \right)^{4j/3}$$

Level 13. Joint work with Dongxi Ye

$$(x_1, x_2, \ldots, x_m; q)_\infty = \prod_{j=0}^{\infty} (1 - x_1 q^j)(1 - x_2 q^j) \cdots (1 - x_m q^j)$$

$$R = R(q) = q \prod_{j=1}^{\infty} (1 - q^j)^{\left(\frac{j}{13}\right)} = q \frac{(q, q^3, q^4, q^9, q^{10}, q^{12}; q^{13})_\infty}{(q^2, q^5, q^6, q^7, q^8, q^{11}; q^{13})_\infty}$$

$$\mathscr{r} = r^5(q) = q \prod_{j=1}^{\infty} (1 - q^j)^{5\left(\frac{j}{5}\right)} = q \frac{(q, q^4; q^5)_\infty^5}{(q^2, q^3; q^5)_\infty^5}$$

$$\frac{1}{R} - 3 - R = \frac{1}{q} \prod_{j=1}^{\infty} \frac{(1 - q^j)^2}{(1 - q^{13j})^2}$$

$$\frac{1}{\mathscr{r}} - 11 - \mathscr{r} = \frac{1}{q} \prod_{j=1}^{\infty} \frac{(1 - q^j)^6}{(1 - q^{5j})^6}$$

Factorizations: Rogers-Ramanujan continued fraction

$$\frac{1}{\mathcal{r}} - 11 - \mathcal{r} = \frac{1}{q}\prod_{j=1}^{\infty}\frac{(1-q^j)^6}{(1-q^{5j})^6}$$

$$\frac{1}{\mathcal{r}} - 11 - \mathcal{r} = \left(\frac{1}{\sqrt{\mathcal{r}}} - \alpha^5\sqrt{\mathcal{r}}\right)\left(\frac{1}{\sqrt{\mathcal{r}}} - \beta^5\sqrt{\mathcal{r}}\right)$$

$$\frac{1}{\sqrt{\mathcal{r}}} - \alpha^5\sqrt{\mathcal{r}} = \frac{1}{(q^5 s)^{1/12}}\prod_{j=1}^{\infty}\frac{1}{(1-\zeta q^j)^5(1-\zeta^4 q^j)^5}$$

$$\frac{1}{\sqrt{\mathcal{r}}} - \beta^5\sqrt{\mathcal{r}} = \frac{1}{(q^5 s)^{1/12}}\prod_{j=1}^{\infty}\frac{1}{(1-\zeta^2 q^j)^5(1-\zeta^3 q^j)^5}.$$

$$\alpha = \frac{1-\sqrt{5}}{2}, \quad \beta = \frac{1+\sqrt{5}}{2}, \quad \zeta = \exp(2\pi i/5), \quad s = \frac{\eta_5^6}{\eta_1^6}$$

Factorizations: level 13

$$\frac{1}{R} - 3 - R = \frac{1}{q}\prod_{j=1}^{\infty}\frac{(1-q^j)^2}{(1-q^{13j})^2}$$

$$\frac{1}{R} - 3 - R = \left(\frac{1}{\sqrt{R}} - \gamma\sqrt{R}\right)\left(\frac{1}{\sqrt{R}} - \delta\sqrt{R}\right)$$

$$\frac{1}{\sqrt{R}} - \gamma\sqrt{R} = \frac{1}{(qS)^{1/4}} \times \frac{1}{(\xi q, \xi^3 q, \xi^4 q, \xi^9 q, \xi^{10} q, \xi^{12} q; q)_\infty}$$

$$\frac{1}{\sqrt{R}} - \delta\sqrt{R} = \frac{1}{(qS)^{1/4}} \times \frac{1}{(\xi^2 q, \xi^5 q, \xi^6 q, \xi^7 q, \xi^8 q, \xi^{11} q; q)_\infty}.$$

$$\gamma = \frac{3-\sqrt{13}}{2}, \quad \delta = \frac{3+\sqrt{13}}{2}, \quad \xi = \exp(2\pi i/13), \quad S = \frac{\eta_{13}^2}{\eta_1^2}$$

$$
\begin{aligned}
&(q^2, q^5, q^6, q^7, q^8, q^{11}, q^{13}, q^{13}; q^{13})_\infty \\
&\quad -\gamma\, q\, (q, q^3, q^4, q^9, q^{10}, q^{12}, q^{13}, q^{13}; q^{13})_\infty \\
&\quad = (\xi^2 q, \xi^5 q, \xi^6 q, \xi^7 q, \xi^8 q, \xi^{11} q, q, q; q)_\infty
\end{aligned}
$$

$$
\begin{aligned}
&(q^2, q^5, q^6, q^7, q^8, q^{11}, q^{13}, q^{13}; q^{13})_\infty \\
&\quad -\delta\, q\, (q, q^3, q^4, q^9, q^{10}, q^{12}, q^{13}, q^{13}; q^{13})_\infty \\
&\quad = (\xi q, \xi^3 q, \xi^4 q, \xi^9 q, \xi^{10} q, \xi^{12} q, q, q; q)_\infty .
\end{aligned}
$$

$$
\gamma = \frac{3-\sqrt{13}}{2}, \quad \delta = \frac{3+\sqrt{13}}{2}, \quad \xi = \exp(2\pi i/13)
$$

Weight 2 modular functions

$$q\frac{d}{dq}\log\left(\frac{r}{1-11r-r^2}\right)=\sum_{n=0}^{\infty}a(n)\left(\frac{r(1-11r-r^2)}{(1+r^2)^2}\right)^n$$

$$a(n)=\binom{2n}{n}\sum_{j=0}^{n}\binom{n}{j}^2\binom{n+j}{j}.$$

The coefficients $a(n)$ satisfy a 3-term recurrence relation.

$$q\frac{d}{dq}\log\left(\frac{R}{1-3R-R^2}\right)=\sum_{n=0}^{\infty}A(n)\left(\frac{R(1-3R-R^2)}{(1+R^2)^2}\right)^n$$

The coefficients $A(n)$ satisfy a 6-term recurrence relation.

Degree 13 hypergeometric transformation formulas

$$\frac{{}_2F_1\left(\frac{1}{12},\frac{5}{12};1;\frac{1728x}{(1+5x+13x^2)(1+247x+3380x^2+15379x^3+28561x^4)^3}\right)}{\sqrt[4]{1+247x+3380x^2+15379x^3+28561x^4}}$$

$$=\frac{{}_2F_1\left(\frac{1}{12},\frac{5}{12};1;\frac{1728x^{13}}{(1+5x+13x^2)(1+7x+20x^2+19x^3+x^4)^3}\right)}{\sqrt[4]{1+7x+20x^2+19x^3+x^4}}$$

$$\frac{{}_3F_2\left(\frac{1}{6},\frac{5}{6},\frac{1}{2};1,1;\frac{1728x}{(1+5x+13x^2)(1+247x+3380x^2+15379x^3+28561x^4)^3}\right)}{\sqrt{1+247x+3380x^2+15379x^3+28561x^4}}$$

$$=\frac{{}_3F_2\left(\frac{1}{6},\frac{5}{6},\frac{1}{2};1,1;\frac{1728x^{13}}{(1+5x+13x^2)(1+7x+20x^2+19x^3+x^4)^3}\right)}{\sqrt{1+7x+20x^2+19x^3+x^4}}.$$

Other levels similar to 5 and 13: $(\ell - 1) \mid 24$

If $(\ell - 1) \mid 24$, let $f(\ell) = \dfrac{\ell P(q^\ell) - P(q)}{\ell - 1}$.

$$f(\ell) = \sum_{j=0}^{\infty} A_\ell(j) x_\ell^j$$

where $A_\ell(j)$ satisfies a recurrence relation, of order given by:

ℓ	2	3	4	5	7	9	13	25
order	2	2	2	3	3	3	6	9

Level 11. Joint work with J. Ge and D. Ye

$$f(3) = \frac{3P(q^3) - P(q)}{2} = \left(\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{j^2+jk+k^2} \right)^2$$

$$f(7) = \frac{7P(q^7) - P(q)}{6} = \left(\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{j^2+jk+2k^2} \right)^2$$

$$f(11) = \left(\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} q^{j^2+jk+3k^2} \right)^2$$

$$f(\ell) = \sum_{j=0}^{\infty} A_\ell(j) \left(\frac{\eta_1^2 \eta_\ell^2}{f(\ell)} \right)^{12j/(\ell+1)}, \qquad \ell = 3, 7, 11$$

$$(j+1)^3 A_{11}(j+1) = 2(2j+1)(5j^2+5j+2)A_{11}(j) \\ -8j(7j^2+1)A_{11}(j-1) + 22j(j-1)(2j-1)A_{11}(j-2).$$

Levels 14 and 15. Joint work with Dongxi Ye

Similar results, because:

$$\sum_{d|14} d = 1+2+7+14 = 24 \qquad \sum_{d|15} d = 1+3+5+15 = 24$$

Cubic transformation of a level 5 function

$$f(x) = \sum_{n=0}^{\infty} \binom{2n}{n} \left\{ \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j} \right\} x^n = \sum_{n=0}^{\infty} c_n x^n$$

$$\frac{1}{1+9x+27x^2} f\left(\frac{x}{(1+9x+27x^2)^2} \right)$$
$$= \frac{1}{1+3x+3x^2} f\left(\frac{x^3}{(1+3x+3x^2)^2} \right)$$

Quintic transformation of a level 3 function

$$g(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad a_0 = 1$$

$$(n+1)^3 a_{n+1} = (2n+1)(7n^2+7n+3)a_n - n(29n^2+4)a_{n-1} + 30n(n-1)(2n-1)a_{n-2}$$

$$x = \frac{v}{(1+3v)^2} = \frac{w}{1+w-w^2} \quad \text{near} \quad x = 0$$

$$\begin{aligned} xg(x) &= \frac{vw}{3v(1+w^2)-2w(1-9v^2)}\, {}_3F_2\left(\begin{matrix}\frac{1}{3},\ \frac{1}{2},\ \frac{2}{3}\\ 1,\ 1\end{matrix}; \frac{108v^3w^2}{(w^2+27v^3)^2}\right) \\ &= \frac{5vw}{3v(1+w^2)+2w(1-9v^2)}\, {}_3F_2\left(\begin{matrix}\frac{1}{3},\ \frac{1}{2},\ \frac{2}{3}\\ 1,\ 1\end{matrix}; \frac{108v^3w^2}{(1+27v^3w^2)^2}\right). \end{aligned}$$

Jacobi, level 4

$$\left(\sum_{j=-\infty}^{\infty} q^{j^2}\right)^4 = \left(\sum_{j=-\infty}^{\infty} (-1)^j q^{j^2}\right)^4 + \left(\sum_{j=-\infty}^{\infty} q^{(j+\frac{1}{2})^2}\right)^4$$

Level 15 analogue

$$\left(\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{j^2+jk+k^2}\right)^2 + 5\left(\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{5j^2+5jk+5k^2}\right)^2$$

$$= 3\left(\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{j^2+jk+4k^2}\right)^2 + 3\left(\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} q^{2j^2+jk+2k^2}\right)^2$$

Other levels

Level 10

$$t(k) = \sum_{j=0}^{k} \binom{k}{j}^4, \text{ Franel, 1895, four-term recurrence}$$

$$\kappa = r(q)r^2(q^2) = q\prod_{j=1}^{\infty} \frac{(1-q^{10j-9})(1-q^{10j-8})(1-q^{10j-2})(1-q^{10j-1})}{(1-q^{10j-7})(1-q^{10j-6})(1-q^{10j-4})(1-q^{10j-3})}$$

Level 12

$$\kappa_{12} = q\prod_{j=1}^{\infty} \frac{(1-q^{12j-11})(1-q^{12j-1})}{(1-q^{12j-5})(1-q^{12j-7})}$$

The coefficients satisfy a six-term recurrence relation.

There are also results for levels 18, 20, 24, 32

Summary

Table: Order of recurrence relations for level ℓ

ℓ	1..4	5..10	11	12, 13	14..16	18	20, 24, 32	25
order	2	3	4	6	4	3	4	9

Levels 1, 2, 3: Ramanujan's "alternative theories"

Level 4: Jacobi

Levels 5, 6, 8, 9: Zagier's sporadic sequences

Level 5: Rogers-Ramanujan cont. frac. Weight 1: Apéry $\zeta(2)$

Level 6: 3 theories, one involves the cubic continued fraction

Weight 2: Apéry $\zeta(3)$, Domb, Almkvist-Zudilin numbers

Levels 2, 3, 4, 5, 7, 9, 13, 25: $(\ell - 1)|24$ share a common theory

Levels 3, 7, 11: another common theory

Levels 14, 15: very similar theories

Ramanujan's series for $1/\pi$

Ramanujan-Gosper, level 2, degree 29

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \binom{2k}{k}^2 \binom{4k}{2k} \frac{(1103 + 26390k)}{396^{4k}}.$$

$$\left(\frac{\eta_1^2 \eta_2^2}{2P(q^2) - P(q)} \right)^4 \Bigg|_{q=\exp(-2\pi\sqrt{29/2})} = \frac{1}{396^4}$$

Another example, level 11, degree 17

$$\frac{1}{\pi} = \frac{\sqrt{11}}{484} \sum_{k=0}^{\infty} (-1)^k \, c(k) \frac{(67 + 221k)}{44^k}$$

where $c(k)$ satisfies a 4-term recurrence relation.

$$\left(\frac{\eta_1 \eta_{11}}{\sum_j \sum_k q^{j^2+jk+3k^2}} \right)^2 \Bigg|_{q=-\exp(-\pi\sqrt{17/11})} = \frac{-1}{44}$$

Another example, with an "80" in it

Zagier's sporadic sequence: $(a, b, c) = (10, 3, -9)$

Level: $\ell = 6$

Degree: $N = 5$

$$\frac{1}{\pi} = \frac{1}{3^{7/2}} \sum_{k=0}^{\infty} \binom{2k}{k} \left\{ \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j} \right\} \frac{(13 + 80k)}{18^{2k}}$$

$$x := \left(\frac{4\eta_1\eta_2\eta_3\eta_6}{6P_6 - 3P_3 + 2P_2 - P_1} \right)^2 \Bigg|_{q=\exp(-2\pi\sqrt{N/\ell})} = \frac{1}{18^2}$$

$$\sqrt{\frac{N}{\ell}} \times 2\sqrt{1 - 4ax - 16cx^2} = \frac{80}{3^{7/2}}.$$

The end!