Title: Chromatic quasisymmetric functions and regular semisimple Hessenberg varieties.

Abstract: We call a weakly increasing sequence $m = (m_1, \ldots, m_n)$ of positive integers a *Hessenberg vector* if $i \leq m_i \leq n$ for all i. Given a Hessenberg vector m and an $n \times n$ matrix s, we define a graph G_m and a variety $X(m, s)$ as follows: the graph G_m has vertex set $\{1, \ldots, n\}$ and edges all ij such that $i < j \leq m_i$. The *Hessenberg variety* $V(m, s)$ is the subvariety of the flag variety in $\mathbb{C}^n$ consisting of all flags $0 < V_1 < \ldots < V_n = \mathbb{C}^n$ such that $sV_i \leq V_{m_i}$ for all i. We consider only the case where s is diagonalizable with n distinct eigenvalues, that is, a regular semisimple matrix.

Certain classes of Hessenberg varieties have been of interest to geometers for some time. The study of regular semisimple Hessenberg varieties was initiated in papers DeMari–Shayman and DeMari–Procesi–Shayman. Tymoczko observed that the action of $C_{GL_n(\mathbb{C})}(s)$ on $V(m, s)$ allows one to apply the theory of torus actions developed by Goresky–Kottwitz–MacPherson, from which one obtains a representation of the symmetric group S_n on each cohomology group of $V(m, s)$. The cohomology of $V(m, s)$ is concentrated in even degrees.

Stanley defined, for any finite graph G, the chromatic symmetric function $X_G(\mathbf{x})$ in variables $x_1, x_2, \ldots$. In their study of a problem about immanants raised by Stembridge, Stanley–Stembridge conjectured that if G is the incomparability graph of a $3 + 1$-free poset P, then $X_G(\mathbf{x})$ is a non-negative integer combination of elementary symmetric functions (an *e-positive* symmetric function). Guay-Paquet showed that to prove the Stanley–Stembridge conjecture, it suffices to show that $X_G(\mathbf{x})$ is e-positive when P is both $3+1$-free and $2+2$-free, in which case G is isomorphic with G_m for some Hessenberg vector m.

In joint work with Wachs, we introduced a graded version $X_G(\mathbf{x}; t)$ of Stanley's chromatic symmetric function. The power series $X_G(\mathbf{x}; t)$ is not symmetric in general, but is a quasisymmetric function. One can consider $X(\mathbf{x}; t)$ to be a polynomial in t with coefficients in the ring of quasisymmetric functions.

We showed that if m is a Hessenberg vector, then $X_{G_m}(\mathbf{x}; t)$ is symmetric. In fact, our conjecture that the Frobenius characteristic of the representation of S_n on $H^{2j}(X(m, s))$ is the coefficient of t^j in $\omega(X_{G_m}(\mathbf{x}; t))$ was proved by Brosnan–Chow. A different proof was given by Guay-Paquet. (Here ω is the involution on the ring of symmetric functions mapping elementary symmetric functions to complete

homogeneous symmetric functions.) As we have determined the decomposition of $X_{G_m}(\mathbf{x};t)$ into Schur functions (generalizing a result of Gasharov for $X_{G_m}(\mathbf{x})$), and Athanasiaids has proved a conjecture of ours giving a formula for the decomposition of $X_{G_m}(\mathbf{x};t)$ into power sum symmetric functions, the Brosnan–Chow theorem tells us the irreducible decomposition and character values of the representation of S_n on the cohomology of $X(m,s)$, thus answering a question of Tymoczko. The Brosnan–Chow theorem also makes possible a geometric approach to proving the Stanley–Stembridge conjecture, which has yet to succeed.

I will discuss all of the work described above. If time permits, I will also discuss connections with representations of type A Hecke algebras, found in work of Haiman and Clearman–Hyatt–Shelton–Skandera and with MacDonald polynomials, found in work of Haglund-Wilson.