SOME PROBLEMS

DENNIS STANTON

Contents

1. Alternating sign matrices 2
2. Eigenvalues of graphs 4
3. Ranks and Cranks 4
4. The Borwein and Bressoud conjectures 6
5. Assorted $q$-binomial questions 8
6. Finite fields 10
7. Rogers-Ramanujan identities 12
8. Other questions 13
References 15

Some of these problems do not originate with me.
1. Alternating sign matrices

An $n \times n$ alternating sign matrix $A$ is an $n \times n$ matrix, with entries $0, \pm 1$, whose row and column sums are 1, and non-zero entries in each row and column alternate in sign, see [13].

The number of $n \times n$ alternating sign matrices is known [48] to be

$$ASM(n) = \prod_{k=0}^{n-1} \frac{(3k + 1)!}{(n + k)!}.$$  

This sequence starts $1, 2, 7, 42, 429, 7436, \cdots$.

It is known that $ASM(n)$ is equal to two other numbers

1. the number of totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box, $TSSCPP(n)$,
2. the number of descending plane partitions whose largest part is at most $n$, $DPP(n)$.

A descending plane partition (DPP) [3] is a column strict tableau of shifted shape, decreasing along rows, and strictly decreasing down columns, such that the lead element of a row is greater than the number of elements in a that row. Here are the 7 descending plane partitions counted by $DPP(3)$

$$\emptyset, 2, 3, 3 2, 3 1, 3 3, 3 3 2.$$  

Open Problem 1.1. Find a bijection between the elements counted by $ASM(n)$ and those counted by $TSSCPP(n)$ or $DPP(n)$.

It is not even known how to do this via the involution principle [22].

It is known [42] that the values of $n$ for which $ASM(n)$ is odd are

$$n = \sum_{t/2 \geq k \geq 0} 2^{t-2k} + \{1 \text{ if } t \text{ is odd}\}.$$  

The first few values are $1, 3, 5, 11, 21, 43, 85, \cdots$.

Open Problem 1.2. Find a Franklin type involution which proves that $ASM(n)$ is even when $n$ avoids the above sequence.

Open Problem 1.3. Find a statistic on a subset of permutations, $T_n \subset S_n$, $stat(w)$, such that

$$ASM(n) = \sum_{w \in T_n} 2^{stat(w)}.$$  

Andrews conjectured [3], and Mills, Robbins, and Rumsey proved [28], that the generating function for $DPP(n)$ is

$$DPP(n, q) = \prod_{k=0}^{n-1} \frac{(3k + 1)!_q}{(n + k)!_q},$$  

for example

$$DPP(3, q) = \frac{7!_q * 4!_q}{3!_q * 4!_q * 5!_q} = \frac{7_q * 6_q}{2_q * 3_q} = (1 - q + q^2)(1 + q + q^2 + q^3 + q^4 + q^5 + q^6) = 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8.$$  

Open Problem 1.4. Find a statistic on $ASM(n)$ whose generating function is $DPP(n,q)$.

It is easy to see that
\[
DPP(\infty, q) = \lim_{n \to \infty} DPP(n, q) = \frac{1}{(1 - q^2)(1 - q^3)(1 - q^4)(1 - q^5)^2(1 - q^6)^2 \cdots}
\]
\[
= \prod_{k=1}^{\infty} \frac{(1 - q^{3k-1})^{-k}(1 - q^{3k})^{-k}(1 - q^{3k+1})^{-k}}{1 - q^{k+1}^{-1}}.
\]

This infinite product may be rewritten using Cauchy’s formula for Schur functions as
\[
DPP(\infty, q) = \sum_{\lambda} s_{\lambda}(q^2, q^3, q^4, \cdots) s_{\lambda}(1, q^3, q^6, \cdots).
\]

Open Problem 1.5. Find a weight preserving bijection between $DPP(\infty)$ and pairs of column strict tableaux of the same shape which proves (1).

Open Problem 1.6. Find a weight preserving bijection between $ASM(\infty)$ and pairs of column strict tableaux of the same shape which proves (1). Restrict this bijection to find a bijection between $DPP(n)$ and $ASM(n)$, also a $q$-statistic for $ASM(n)$.

Let $G$ be the cyclic group of order 4 which acts by rotations on the set of $n \times n$ alternating sign matrices. It is known [36] that $(ASM(n), DPP(q), G)$ is an example of the cyclic sieving phenomenon. Thus $DPP(i)$ is the number of $n \times n$ alternating sign matrices fixed under a 90 degree rotation.

Open Problem 1.7. Find an insightful (non computational) proof that $(ASM(n), DPP(n,q), G)$ is an example of the CSP.

Tom Sundquist [43] defined, for positive integers $n$ and $p$,
\[
A(n, p; q) = \prod_{k=0}^{n-1} \frac{(np + k)! (p + 1)!}{q^{(p + 1)k} ((p + 1)k)! q^{(p + 1)! q^{(p + 1)k}^{-1}}}
\]
\[
= q^{-P} s_{(p \delta_n)}(1, q, \cdots, q^{np-1}) s_{p \delta_n}(1, q, \cdots, q^{n-1}).
\]

where
\[
\delta_n = (n - 1, n - 2, \cdots, 0), \quad P = \binom{p}{2} \sum_{i=1}^{n-1} i^2.
\]

Sundquist proved this was always a polynomial in $q$ with integer coefficients, but did not prove positivity.

Open Problem 1.8. Prove $A(n, p; q) \in \mathbb{N}[q]$ if $n$ and $p$ are positive integers and what does $A(n, p; q)$ count?
Note that the $DPP$ and Catalan are both special cases, so positivity is known.  
\[ A(n, 2; q) = DPP(n, q) \]
\[ A(2, p; q) = \frac{1}{[p+1]q} \left[ \begin{array}{c} 2p \\ p \end{array} \right] = Cat_p(q) \]

Sundquist also gives a combinatorial interpretation for $A(\infty, p; q)$.  For $p = 2$, Jessica Striker has noted that this result should be the following.

**Proposition 1.9.** $DPP(\infty, q)$ is the generating function for all plane partitions $T$ of the following type.  For any $i$, the elements of the $i^{\text{th}}$ column are $1, 2, \ldots, i - 1$ or $i + 1, i + 4, i + 7, \ldots$.

**Open Problem 1.10.** What restriction on the plane partitions $T$ in Proposition 1.9 allows the generating function $DPP(n, q)$?  Find a bijection between this class and any of the three known ASM$(n)$-equivalent objects.

## 2. Eigenvalues of Graphs

Let $G$ be a finite simple graph with $n$ vertices $\{v_1, \cdots, v_n\}$.  The Laplacian matrix $L(G)$ is an $n \times n$ matrix whose entries are

\[ L(G)_{ij} = \begin{cases} 
\text{deg}(v_i) & \text{if } i = j, \\
-1 & \text{if } i \neq j \text{ and } v_i - v_j \text{ is an edge}, \\
0 & \text{if } i \neq j \text{ and } v_i - v_j \text{ is not an edge}
\end{cases} \]

It is known that $L(G)$ is singular, diagonalizable, and positive semidefinite.  So one eigenvalue of $L(G)$ is 0, and let $\lambda_1, \cdots, \lambda_{n-1}$ be the remaining non-negative eigenvalues.  It is known that

\[ \lambda_1 \ast \cdots \ast \lambda_{n-1} = e_{n-1}(\lambda_1, \cdots, \lambda_{n-1}) \]

is the number of rooted spanning trees of $G$.  Moreover the combinatorial interpretation of the coefficients of the characteristic polynomial of $L(G)$ shows that

\[ e_{n-k}(\lambda_1, \cdots, \lambda_{n-1}) \]

is the number of spanning forests of $G$ consisting of $k$ rooted trees.

**Open Problem 2.1.** What is the combinatorial interpretation of the Schur function

\[ s_\mu(\lambda_1, \cdots, \lambda_{n-1})? \]

This is a non-negative integer, because of the Jacobi-Trudi identity and the non-negativity of the eigenvalues.

## 3. Ranks and Cranks

The Dyson rank [15] of an integer partition $\lambda = (\lambda_1, \lambda_2, \cdots)$

\[ \text{rank}(\lambda) = \lambda_1 - \lambda_1' \]

(largest part - number of parts) proves the Ramanujan congruences

\[ p(5n + 4) \equiv 0 \mod 5, \quad p(7n + 5) \equiv 0 \mod 7 \]

by considering the rank modulo 5 and 7.  No one knows bijections for these rank classes.
The generating function for the rank polynomial is known to be
\[
\sum_{n=0}^{\infty} \text{rank}_n(z) q^n = \sum_{n=0}^{\infty} \frac{q^n}{(z; q)_n(q/z; q)_n}
\]

The rank generating function \( \text{rank}_{5n+4}(z) \) for partitions of \( 5n+4 \) does have an explicit factor of 5, but not positively. For example
\[
\text{rank}_4(z) = 1 + z - 3 + z^2 - z^3 + z^4 = (1 + z + z^2 + z^3 + z^4) * (1 - z + z^2)/z^3,
\]
\[
\text{rank}_{14}(z) = (1 + z + z^2 + z^3 + z^4) * p(z)/z^{13}
\]
where \( p(z) \) is an irreducible polynomial of degree 22 which has negative coefficients.

For an explicit 5-cycle which would be a rank class bijection, one would expect the factor \( 1 + z + z^2 + z^3 + z^4 \) times a positive Laurent polynomial in \( z \). Here is a conjectured modification that does this.

**Definition 3.1.** For \( n \geq 2 \) let
\[
M_{\text{rank}}_n(z) = \text{rank}_n(z) + (z^{n-2} - z^{n-1} + z^{2-n} - z^{1-n}).
\]

**Conjecture 3.2.** For \( n \geq 0 \),
\[
M_{\text{rank}}_{5n+4}(z)/(1 + z + z^2 + z^3 + z^4)
\]
is a non-negative Laurent polynomial in \( z \). Also
\[
M_{\text{rank}}_{7n+5}(z)/(1 + z + z^2 + z^3 + z^4 + z^5 + z^6)
\]
is a non-negative Laurent polynomial in \( z \).

This conjecture says that the rank definition only needs to be changed for \( \lambda = n, 1^n \) to have the “correct” symmetry. I do not know a modification which will also work modulo 11. Frank Garvan has verified Conjecture 3.2 for \( 5n + 4 \leq 1000 \) and \( 7n + 5 \leq 1000 \).

The Andrew-Garvan [5] crank of a partition \( \lambda \) is
\[
\text{AGcrank}(\lambda) = \begin{cases} 
\lambda_1 & \text{if } \lambda \text{ has no 1's} \\
\mu(\lambda) - (#1's \text{ in } \lambda) & \text{if } \lambda \text{ has at least one 1,}
\end{cases}
\]
where \( \mu(\lambda) \) is the number of parts of \( \lambda \) which are greater than the number of 1’s of \( \lambda \). For example
\[
\text{AGcrank}(1111) = 0 - 4, \text{ AGcrank}(211) = 0 - 2, \text{ AGcrank}(22) = 2 - 0 \\
\text{AGcrank}(31) = 1 - 1, \text{ AGcrank}(4) = 4 - 0
\]

The generating function of the AGcrank over all partitions of \( n \) is \( \text{AGcrank}_n(z) \).

For example
\[
\text{AGcrank}_4(z) = z^{-4} + z^{-2} + z^2 + z^0 + z^4.
\]

The generating function for the AGcrank polynomial is known to be (after modifying \( \text{AGcrank}_4(z) \))
\[
\sum_{n=0}^{\infty} \text{AGcrank}_n(z) q^n = \frac{(q; q)_\infty}{(zq; q)_\infty(q/z; q)_\infty}
\]

**Open Problem 3.3.** Show
\[
\text{AGcrank}_{5n+4}(z) = (1 + z^2 + z^4 + z^6 + z^8) * (a \text{ positive Laurent polynomial in } z).
\]
Frank Garvan has verified Open Problem 3.3 for $5n + 4 \leq 1000$.
Ramanujan factored the first 21 AGcrank polynomials, $\lambda_n = AGcrank_n(a)$, see the paper of Berndt, Chan, Chan and Liaw [9, p. 12]. Ramanujan found the factor $\rho_5 = z^4 + z^{-4} + z^2 + z^{-2} + 1$ for $n = 4, 9, 14, 19$ but the other factors did not always have positive coefficients. For example Ramanujan had
\[
AGcrank_{14}(z) = (z^4 + z^2 + 1 + z^{-2} + z^{-4}) * \rho_9 * (a_5 - a_3 + a_1 + 1),
\]
where
\[
\rho_9 * (a_5 - a_3 + a_1 + 1) = (z^2 + z^{-2} + 1)(z^3 + z^{-3} + 1) * (z^5 + z^{-5} - z^3 - z^{-3} + z + z^{-1} + 1)
\]
\[
= 3 + 1/z^{10} + 1/z^7 + 1/z^6 + 1/z^5 + 2/z^4 + 2/z^3 + 2/z^2 + 2/z
\]
\[+ 2z + 2z^2 + 2z^3 + 2z^4 + z^5 + z^6 + z^7 + z^{10}.
\]

A modified version of the AGcrank works for modulo 5, 7, and 11, with only the values at partitions $n, 1^n$ changed.

**Definition 3.4.** For $n \geq 2$ let
\[
MAGcrank_{n,a}(z) = AGcrank_n(z) + (z^{n-a} - z^n + z^{a-n} - z^{-n}).
\]

**Conjecture 3.5.** The following are non-negative Laurent polynomials in $z$
\[
MAGcrank_{5n+4,5}(z)/(1 + z + z^2 + z^3 + z^4),
MAGcrank_{7n+5,7}(z)/(1 + z + z^2 + z^3 + z^4 + z^5 + z^6),
MAGcrank_{11n+6,11}(z)/(1 + z + z^2 + z^3 + z^4 + z^5 + z^6 + z^7 + z^8 + z^9 + z^{10}).
\]

Frank Garvan has verified Conjecture 3.5 for $tn + r \leq 1000$.

The 5corecrank (see [23, 1990]) may be defined from the integer parameters $(a_0, a_1, a_2, a_3, a_4)$ involved in the 5-core of a partition $\lambda$. Its generating function for partitions of $5n + 4$ is
\[
\sum_{n=0}^{\infty} q^{n+1} \sum_{\lambda \vdash 5n+4} z^{5\text{corecrank}(\lambda)} = \frac{1}{(q; q)_\infty} \sum_{\vec{a} \in \mathbb{Z}^4} q^{Q(a)} z^{\sum_{i=1}^{4} i a_i},
\]
where
\[
Q(a) = \sum_{i=0}^{4} a_i^2 - \sum_{i=0}^{4} a_ia_{i+1}, \quad a_5 = a_0.
\]

Frank Garvan also noted the following version of the previous conjectures holds for the 5corecrank for $n \leq 100$, and $n \leq 8$, see [7]. Ken Ono [33], in work with Bringmann and Rolen, has established the first statement.

**Conjecture 3.6.** The following are non-negative Laurent polynomial in $z$
\[
5\text{corecrank}_{5n+4}(z)/(1 + z + z^2 + z^3 + z^4),
5\text{corecrank}_{5n+4,j}(z)/(1 + z + z^2 + z^3 + z^4) \quad \text{when restricted to } BGcrank = j.
\]

4. The Borwein and Bressoud conjectures

The Borwein conjecture ([1], proven by Chen Wang [44] in 2019, see also [10]) is the following positivity conjecture.
Let
\[
(q; q^3)_n(q^2; q^3)_n = A_n(q^3) - qB_n(q^3) - q^2 C_n(q^3)
\]
for polynomials $A_n(q), B_n(q), C_n(q)$. Then $A_n(q), B_n(q), C_n(q)$ have non-negative coefficients.

There are explicit alternating forms for these polynomials

$$A_n(q) = \sum_k \left[ \frac{2n}{n - 3k} \right]_q (-1)^k q^{(9k^2 - k)/2}$$

$$B_n(q) = \sum_k \left[ \frac{2n}{n - 3k - 1} \right]_q (-1)^k q^{(9k^2 + 5k)/2}$$

$$C_n(q) = \sum_k \left[ \frac{2n}{n - 3k + 1} \right]_q (-1)^k q^{(9k^2 - 7k)/2}$$

Note that $A_n(1) = 2 \cdot 3^{n-1}, B_n(1) = 3^{n-1} = C_n(1)$. The $n = \infty$ case follows from the Jacobi triple product

$$A_\infty(q) = \frac{(q^4; q^3; q^9)_\infty}{(q; q)_\infty},$$

$$B_\infty(q) = \frac{(q^7; q^2; q^9)_\infty}{(q; q)_\infty},$$

$$C_\infty(q) = \frac{(q^8; q^1; q^9)_\infty}{(q; q)_\infty}.$$

**Open Problem 4.1.** Identify finite subsets of partitions whose parts are restricted modulo 9 via (2) whose generating functions are $A_n(q), B_n(q), C_n(q)$.

It is known that the hook difference polynomials do have positive coefficients and count certain partitions which lie inside a rectangle (see [4]).

Let $N, M, i, K, \alpha, \beta$ be positive integers such that

$$\alpha + \beta < K, \quad -i + \beta \leq N - M \leq K - i - \alpha.$$  

Then the hook difference polynomials are

$$D_{K,i}(N, M, \alpha, \beta) = \sum_\lambda q^{\lambda(K\lambda + i)(\alpha + \beta) - K\beta\lambda} \left[ \frac{N + M}{N - K\lambda} \right]_q$$

$$- \sum_\lambda q^{\lambda(K\lambda - i)(\alpha + \beta) - K\beta\lambda + \beta i} \left[ \frac{N + M}{N - K\lambda + i} \right]_q$$

A special case is

$$D_{2k,k}(N, N, \alpha, \beta) = \sum_s (-1)^s \left[ \frac{2N}{N - ks} \right]_q q^{ks(s+1)(\alpha + \beta)/2 - \beta ks}$$

for

$$\alpha + \beta < 2k, \quad -k + \beta \leq 0 \leq k - \alpha.$$  

The Borwein polynomial $A_n(q) = D_{6,3}(N, N, 4/3, 5/3)$.

**Open Problem 4.2.** What is the combinatorial meaning of the rational parameters $\alpha = 4/3, \beta = 5/3$?

Bressoud [11] investigated this question and formulated a more general conjecture for rational parameters (his Conjecture 6).
Conjecture 4.3. Let $\alpha$ and $\beta$ be positive rational numbers, and let $k > 1$ be an integer such that $ak$ and $\beta k$ are integers. If
\[
1 \leq \alpha + \beta \leq 2k - 1, \quad (\text{with strict inequalities for } k = 2)
\]
\[\beta \leq n - m \leq k - \alpha \]
then $D_{2k,k}(n, m, \alpha, \beta)$ has non-negative coefficients.

There is also a corresponding Borwein type conjecture for special cases of these polynomials (see [11, Conjecture 5]). If $k$ is odd, $1 < a < k/2$, let
\[
(q^a; q^k)_m(q^{k-a}; q^k)_n = \sum_{\nu=(1-k)/2}^{(k-1)/2} (-1)^\nu q^{k(\nu^2 + \nu)/2 - \nu} F_\nu(q^k)
\]
then
\[
F_\nu(q) = G_{2k,k}(n, m, \nu + (k + 1)/2 - a/k, -\nu + (k - 1)/2 + a/k).
\]

Conjecture 4.4. If $a$ is relatively prime to $k$ and $m = n$, then the coefficients of $F_\nu(q)$ are non-negative.

The Borwein conjecture is the case $k = 3$, $a = 1$.

Conjecture 4.4 says that the coefficients of $q^p, p \equiv \nu \mod k$ in
\[
(q^a; q^k)_n(q^{k-a}; q^k)_n
\]
all have sign $(-1)^\nu$.

The refined Borwein conjecture [1, (1.5)] for the coefficients of $z^t$ in
\[
(q, q^2; q^3)_m(zq, zq^2; q^3)_n
\]
has been proven false in general by Yee, see [46].

If $q = 1$ there is polynomial version [26] which replaces the sign $(-1)^s$ in (3) by a Chebychev polynomial.

Theorem 4.5. If $|N - M| \leq k$, then
\[
\sum_s \binom{N + M}{M - ks} \cos(sx)
\]
is a positive polynomial in $1 + \cos(x)$, and thus is positive for a real value of $x$.

Open Problem 4.6. Find a $q$-version of this result which contains Bressoud’s conjecture. See [26].

5. Assorted $q$-binomial questions

In [39, Theorem 1] it was proven that
\[
\frac{1}{n!_q} = \frac{(1 - q)^n}{(q; q)_n}
\]
has alternating power series coefficients.

Open Problem 5.1. What is the algebraic meaning in terms of Koszul duality of this result?
SOME PROBLEMS

A generalization [39, Theorem 2] was given for the alternating behavior of

\[(5) \quad (1 - q)^n \binom{n + k}{k}_q\]

where \(nk\) is even.

Open Problem 5.2. What is the algebraic meaning in terms of Koszul duality of this result?

Open Problem 5.3. Find sign-reversing involutions which prove (4) and (5) have alternating coefficients.

In [20] a polynomial expansion for the \(q\)-binomial coefficient is given

\[\binom{n}{k}_q = \sum_{\omega \in \Omega_{n,k}} q^{s(\omega)} (1 + q)^{t(\omega)}.\]

The set \(\Omega_{n,k}\) is a subset of the set of all words with \(k\) 1’s and \(n - k\) 0’s.

Open Problem 5.4. Is there an apriori definition of \(\Omega_{n,k}, s(\omega), t(\omega)\) using coset representatives or root systems?

Franklin [2, Ex. 13-14, p. 50] had a generalization of the \(q\)-binomial coefficient

\[\binom{m+k}{k}_q\]

being the generating function for partitions with at most \(k\) parts, largest part at most \(m\).

Theorem 5.5. (Franklin) Let \(1 \leq j \leq k\). The generating function for all partitions \(\lambda\) with at most \(k\) parts such that \(\lambda_1 - \lambda_{j+1} \leq m\) is

\[\frac{(1 - q^{m+1}) \cdots (1 - q^{m+j})}{(1 - q) \cdots (1 - q^k)}.\]

This has an inductive proof by a sign-reversing involution.

Open Problem 5.6. What is the analogue of Franklin’s Theorem 5.5 for the MacMahon box theorem? Is there a result for each symmetry class of plane partitions?

The KOH identity [25, 47] for the \(q\)-binomial coefficient

\[(6) \quad \binom{n + k}{k}_q = \sum_{\lambda \vdash k} q^{2n(\lambda)} \prod_{i=0}^{k-1} \left[ \frac{(k - i)n - 2i + d_{k-i} + \sum_{j=0}^{i-1} 2(i - j)d_{k-j}}{d_{k-i}} \right]_q\]

where \(\lambda = 1^{d_1}2^{d_2} \cdots\) is a partition of \(k\), proves unimodality of the \(q\)-binomial coefficient as a polynomial in \(q\). Stanley [38] proved a stronger theorem for Weyl groups, which implies that \((-q; q)_n\) is unimodal polynomial in \(q\).

Open Problem 5.7. Find a KOH-type identity for \((-q; q)_n = \prod_{i=1}^{n}(1 + q^i)\).

The KOH identity was combinatorially proved under the assumption that

\[\binom{N}{s}_q = 0\]
if \( s \geq 0 \) and \( N < 0 \). Macdonald [32] proved a version, called MACKOH, which assumes the definition

\[
\left[ \frac{N}{s} \right]_q = \frac{(1-q^N)(1-q^{N-1})\cdots(1-q^{N-s+1})}{(1-q)(1-q^2)\cdots(1-q^s)}
\]

for all \( s \geq 0 \), and all \( N \). In this version, both sides of (6) are polynomials in \( x = q^n \) of degree \( k \), true for infinitely many \( x \). Thus (KOH) implies (MACKOH).

**Open Problem 5.8.** Find an involution which proves that the (MACKOH) implies the (KOH).

For example, if \( k = 4 \), then the KOH identity is

\[
\binom{n+4}{4} = \binom{4n+1}{1}_q + q^2 \binom{3n-1}{1}_q \binom{n-1}{1}_q + q^3 \binom{2n-2}{1}_q + q^4 \binom{n-2}{1}_q + q^6 \binom{n-2}{2}_q + q^8 \binom{n-2}{4}_q.
\]

What is the involution, assuming (MACKOH), which shows that terms with negative parameters may be dropped?

This appears to be related to the \( M = N \) conjecture in quantum integrable systems [17, Conj. 2.8].

Gessel [24] defined a collection of Super Catalan numbers which are integers

\[
T(m,n) = \frac{(2n)!(2m)!}{m!n!(m+n)!}.
\]

Combinatorial interpretations have been given for \( m \) small or \( n - m \) small, and also signed versions for all \( m, n \) [6].

Sundquist generalized this result by considering

\[
T(a_1, a_2, \cdots, a_k; q) = \frac{\prod_{i=1}^{k} (2^{k-1}a_i)_q}{\prod_{S \subseteq [k]} a_S^q},
\]

where

\[
a_S = \sum_{s \in S} a_s
\]

and the product in the denominator does not include \( S = [k] \). The Super Catalan numbers are the case \( k = 2, a_1 = m, a_2 = n \). He proved \( T(a_1, a_2, \cdots, a_k; q) \) is a polynomial in \( q \). The positivity for the \( q \)-super Catalan numbers was established in [45, Prop. 2].

**Open Problem 5.9.** Prove \( T(a_1, a_2, \cdots, a_k; q) \in \mathbb{N}[q] \) and find a combinatorial interpretation for this generating function. Is there a result for other posets?

6. Finite fields

If the cyclic group of order \( n \) acts on the set of \( k \)-element subsets of \([n] \), the number of orbits is of size \( e \) is

\[
O(n,k,e) = \frac{1}{e} \sum_{d|s|\operatorname{GCD}(k,n)} \mu\left(\frac{n}{d}\right) \binom{n/s}{k/s}.
\]
A $q$-version of this result was given by Drudge [14], [35, Prop. 9.2]. The number of orbits of the Singer cycle $c$ in $GL_n(\mathbb{F}_q)$ on the $k$-dimensional subspaces of size $[e]_q$, $n = de$ is

$$O(n, k, e; q) = \frac{1}{[e]_q d |GCD(k, n)}} \sum \mu(s) \left[\frac{n}{s}\right]_{q, n}.$$  

The special case $e = n$ is attractive

$$O(n, k, n; q) = \frac{1}{[n]_q} \sum s |GCD(k, n)} \mu(s) \left[\frac{n}{s}\right]_{n, q}.$$ 

In [35, Conj. 10.3] the polynomiality in $q$ of this number is proven, but not positivity.

**Open Problem 6.1.** Prove $O(n, k, n; q) \in \mathbb{N}[q]$ and find a combinatorial interpretation for this generating function.

When $GCD(n, k) = 1$ this is

$$O(n, k, n; q) = \frac{1}{[n]_q} \left[\frac{n}{k}\right]_q$$

and this has a combinatorial interpretation.

In [30, Theorem 1.1], using the complex irreducible characters of $GL_n(\mathbb{F}_q)$, the number of factorizations of the Singer cycle $c$ into $n$ reflections was found to be

$$(q^n - 1)^{n - 1}.$$ 

No simple proof is known. It is the $q$-version of the number of factorizations of an $n$-cycle into transpositions being $n^{n - 2}$.

**Open Problem 6.2.** Find a direct bijection for this result.

More generally [30, Theorem 1.2], the generating function for the number of factorizations, $t(c, \ell)$, of the Singer cycle $c$ into $\ell$ reflections is

$$\sum_{\ell=n}^{\infty} t(c, \ell) x^\ell = (q^n - 1)^{n - 1} \frac{x^n (1 + x[n] q (1 + q^k - q^{k+1}))^{-1}}{1 + x[n] q}.$$ 

**Open Problem 6.3.** Find an apriori reason for the rationality of this generating function. Do the zeros in the denominator have geometric meaning?

Let $S = \mathbb{F}_q[x_1, \ldots, x_n]$ be the polynomial ring on which $GL_n(\mathbb{F}_q)$ naturally acts. Let

$$Q_m = S/<x_1^m, \ldots, x_n^m>.$$ 

on which $GL_n(\mathbb{F}_q)$ also acts. Then [31, Conj. 1.2]

**Conjecture 6.4.** The Hilbert series for the $GL_n(\mathbb{F}_q)$ fixed subalgebra of $Q$ is

$$\sum_{k=0}^{\min(m, n)} t^{(n-k)(q^m - q^k)} \left[\frac{m}{k}\right]_{q, t}.$$ 

In [34] and [40] some results are given for a theory of partitions and plane partitions whose parts sizes are $[n]_q$ instead of integers $n.$

**Open Problem 6.5.** Can these results be extended to other classical partition results?
7. Rogers-Ramanujan identities

There are known polynomial identities which generalize the Rogers-Ramanujan (RR) identities

\[ \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} = \frac{1}{(q, q^4; q^2)_\infty}, \quad \sum_{k=0}^{\infty} \frac{q^{k^2+k}}{(q; q)_k} = \frac{1}{(q^2, q^3; q^5)_\infty}. \]

Schur knew that

\[ D_n(q) = \sum_{k=0}^{(n+1)/2} q^{k^2} \left[ \begin{array}{c} n + 1 - k \\ k \end{array} \right]_q, \quad E_n(q) = \sum_{k=0}^{n/2} q^{k^2+k} \left[ \begin{array}{c} n - k \\ k \end{array} \right]_q \]

had alternative representations

\[ D_{n-1}(q) = \sum_j (-1)^j q^{j(5j+1)/2} \left[ \begin{array}{c} n \\ (n+5j+1)/2 \end{array} \right]_q, \]
\[ E_n(q) = \sum_j (-1)^j q^{j(5j+3)/2} \left[ \begin{array}{c} n + 1 \\ (n+5j+3)/2 \end{array} \right]_q. \]

(Note that both satisfy the $q$-Fibonacci recurrence $p_n = p_{n-1} + q^n p_{n-2}$.) Using the Jacobi-Triple-Product on $D_\infty(q)$, $E_\infty(q)$, one obtains the RR identities. Easily (7) shows that $D_n(q)$ is the generating function for all partitions $\lambda$ whose difference of parts is at least 2, and $\lambda_1 \leq n$.

**Open Problem 7.1.** Which subset of partitions whose parts are restricted modulo 5 do the polynomials in (7) generate?

Bressoud [12, (1.1), (1.3)], [37, Sec. 6] gave another polynomial identity

\[ \sum_{j=0}^{n} q^{j^2} \left[ \begin{array}{c} n \\ j \end{array} \right]_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+1)/2} \left[ \begin{array}{c} 2n \\ n+2j \end{array} \right]_q \]
\[ \sum_{j=0}^{n} q^{j^2+j} \left[ \begin{array}{c} n \\ j \end{array} \right]_q = \sum_{j=-\infty}^{\infty} (-1)^j q^{j(5j+3)/2} \left[ \begin{array}{c} 2n + 1 \\ n+2j+1 \end{array} \right]_q \]

This time the left side generates difference two partitions $\lambda$ with $\text{rank}(\lambda) \leq n-1$.

**Open Problem 7.2.** Which subset of partitions whose parts are restricted modulo 5 do the polynomials in (8) generate?

Ekhad-Tre [16] also found

\[ \sum_{j=0}^{n} q^{j^2} \left[ \begin{array}{c} n \\ j \end{array} \right]_q = \frac{(q; q)_n}{(q; q^2)_n} \sum_{k=-n}^{n} (-1)^k q^{5k^2-5k} \left[ \begin{array}{c} 2n \\ n-k \end{array} \right]_q \]

Sills [37, Sec. 6] also gives polynomial identities for

\[ \sum_{j=0}^{n/2} q^{j^2} \left[ \begin{array}{c} n \\ 2j \end{array} \right]_q. \]
There is a quintic transformation [21, Th. 7.1] which proves the RR identities

\[
\sum_{n=0}^{\infty} q^n (tq)^{2n} = \frac{(t^4 q^9, t^2 q^5; t^4 q^6; q^5)_{\infty}}{(t^2 q^3; q)_{\infty}} \sum_{k=0}^{n} \frac{q^k}{(q; q)_k}
= \frac{(t^4 q^8, t^2 q^3, t^4 q^6; q^5)_{\infty}}{(t^2 q^3; q)_{\infty}} \sum_{k=0}^{n} \frac{q^k}{(q; q)_k}
= \frac{(t^4 q^8, t^2 q^3, t^4 q^6; q^5)_{\infty}}{(t^2 q^3; q)_{\infty}} \sum_{k=0}^{n} \frac{q^k}{(q; q)_k}
\]

The only known proof of (9) uses orthogonal polynomials.

**Open Problem 7.3.** Find another proof of (9). What does it mean combinatorially, or for representations of \(A_1^{(1)}\). Is there a symmetric function version?

One may show that (9) implies a finite rational function identity which is nearly positive

\[
\sum_{k=0}^{n} \frac{q^k}{(q; q)_k} \equiv \frac{q^{5a+b+5(5k+1)+3b}}{(q^5; q^5)_a (q^5; q^5)_b (q^5; q^5)_c (q^5; q^5)_{n-(a+b+c+k+s)}} \sum_{k=0}^{n} q^{5(2s-k)} \frac{(q^5)}{(q^5; q^5)_k}.
\]

No direct bijection for the Rogers-Ramanujan identities is known, although the involution principle of Garsia and Milne [22] was created to give an indirect bijection.

**Open Problem 7.4.** Can this identity be mutated to one with only positive terms, and thereby lead to a direct RR bijection?

Using the Cauchy identity, whose bijective version is RSK, one obtains

\[
\sum_{\lambda} s_{\lambda}(xz, xq^2) s_{\lambda}(q^1, q^6, q^{11}, \ldots) = \frac{1}{(xz^2, xq^2; q^5)_{\infty}}.
\]

Because the Schur function with 2 variables is zero unless the partition has at most two parts, one may rewrite this as

\[
\frac{1}{(xz^2, xq^2; q^5)_{\infty}} = \sum_{N=0}^{\infty} x^N \frac{q^{2N}}{(q^5; q^5)_N} \sum_{b=0}^{[N/2]} q^{b[N-2b+1]} \left( \begin{array}{c} N \\ b \end{array} \right) \left( \begin{array}{c} N \\ b-1 \end{array} \right) q^b.
\]

All terms here are positive. May this be extended to a proof of RR?

In [29] refinements of the Rogers-Ramanujan identities are given by marking parts. These are based upon some sporadic positive rational function identities.

**Open Problem 7.5.** Can these identities be generated via computer algebra? Are they related to decompositions of polytopes, or Hilbert series in commutative algebra?

8. OTHER QUESTIONS

Type \(R_t\) and \(R_{II}\) orthogonal polynomials [27] satisfy the respective recurrence relations

\[
P_n(x) = (x - b_n)P_{n-1}(x) - \lambda_n(x - a_n)P_{n-2}(x),
\]
with the initial conditions
\[ P_0(x) = 1, \quad P_1(x) = x - b_1. \]
and
\[ Q_n(x) = (x - c_n)Q_{n-1}(x) - \lambda_n(x - a_n)(x - b_n)Q_{n-2}(z), \]
with the initial conditions
\[ Q_0(x) = 1, \quad Q_1(x) = x - c_1. \]

There are linear functionals \( L_1 \) and \( L_2 \), defined on the appropriate vector space of rational functions, such that
\[
L_1 \left( x^j P_n(x)/\prod_{k=1}^{n} (x - a_{k+1}) \right) = 0, \quad 0 \leq j < n, \quad L_1(1) = \lambda_1
\]
and
\[
L_2 \left( x^j Q_n(x)/\prod_{k=1}^{n} (x - a_{k+1})(x - b_{k+1}) \right) = 0, \quad 0 \leq j < n.
\]

**Open Problem 8.1.** Develop a Viennnot theory for type \( R_I \) and \( R_{II} \) polynomials.

(Note: September 12, 2019: Jang Soo Kim and I have done this for type \( R_I \).)

**Open Problem 8.2.** Does a \( GL_n(\mathbb{F}_q) \) version of the cycle index generating function easily count involutions in \( GL_n(\mathbb{F}_q) \) (see [18]) or explain the competing \( q \)-versions of the Poisson distribution [19]? Are there separate \( q \)-Central Limit Theorems for the discrete and continuous \( q \)-Hermite polynomials?

A perfect Hamming 1-code is a subset \( S \) of the vertices of the \( n \)-dimensional cube \( X_n = \{0, 1\}^n \) so that the balls of radius one about points of \( S \) are disjoint and cover \( X_n \). Clearly for this to occur, \( n + 1 \) divides \( 2^n \), so \( n \) must be one less than a power of two. Such perfect codes are known to exist.

The \( q \)-analogue of the Hamming scheme is a graph whose vertices are the maximal isotropic subspaces over \( \mathbb{F}_q \), with edges if they overlap maximally. In types \( B_n \) and \( C_n \) it is known that there are \((1 + q)(1 + q^2) \cdots (1 + q^n)\) such vertices, and the ball of radius 1 has size \((1 - q^{n+1})/(1 - q)\). Again the sphere packing condition implies that \( n = 2^k - 1 \) for some \( k \). It is known that such perfect codes exist for \( n = 3 \), but it is unknown for \( n \geq 7 \).

**Open Problem 8.3.** Settle the existence/non-existence question of perfect codes in the association schemes of dual polar spaces of types \( B_n \) and \( C_n \) for \( n = 2^k - 1, k \geq 3 \). See ([41, §8].)

The continuous \( q \)-Hermite polynomials \( p_n(x) \) satisfy
\[
p_{n+1}(x) = xp_n(x) - [n]_q p_{n-1}(x)
\]
while the discrete \( q \)-Hermite polynomials \( r_n(x) \) satisfy
\[
r_{n+1}(x) = xr_n(x) - q^{n-1}[n]_q r_{n-1}(x).
\]

There is another set of \( q \)-Hermite polynomials \( s_n(x) \) which satisfy
\[
s_{n+1}(x) = xs_n(x) - \frac{q^n - q}{q^1 - q^1}s_{n-1}(x).
\]
These polynomials remarkably have linearization formula

\[ s_n s_m = \sum_{k=0}^{\min(m,n)} \ell_{nm}^k s_k, \]

where \( \ell_{nm}^k \) can be proven to be a non-negative polynomial in \( q \). A combinatorial interpretation of the moments for the corresponding indeterminate moment problem is known.

Open Problem 8.4. Find any of the following information about \( s_n(x) \): an explicit formula, generating function, or measure. Is there an Askey scheme with these polynomials at the bottom?

REFERENCES

16 DENNIS STANTON


[34] V. Reiner and D. Stanton, $(q,t)$-analogues and $GL_n(F_q)$. J. Algebraic Combin. 31 (2010), no. 3, 411-454.


School of Mathematics, University of Minnesota, Minneapolis, MN 55455
E-mail address: stanton@math.umn.edu