WHAT IS CYCLIC SIEVING?

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Many finite sets in combinatorics have both cyclic symmetry and a natural generating function. Surprisingly often the generating function evaluated at roots of unity counts symmetry classes. We call this the cyclic sieving phenomenon.

More precisely, let $C$ be a cyclic group generated by an element $c$ of order $n$ acting on a finite set $X$. Given a polynomial $X(q)$ with integer coefficients in a variable $q$, say that the triple $(X, X(q), C)$ exhibits the cyclic sieving phenomenon (CSP) if for all integers $d$, the number of elements fixed by $c^d$ equals the evaluation $X(\zeta^d)$ where $\zeta = e^{2\pi i/n}$. In particular, $X(1)$ is the cardinality of $X$, so that $X(q)$ can be regarded as a generating function for $X$.

In the proto-example, $X$ is the collection of all $k$-element subsets of $\{1, 2, \ldots, n\}$, and $X(q)$ is the renowned $q$-binomial coefficient or Gaussian polynomial

$$X(q) = \binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where $[m]_q := [m]_q[m-1]_q \cdots [2]_q[1]_q$ and $[m]_q := 1 + q + q^2 + \cdots + q^{m-1}$. Let the generator $c$ of $C$ act by cycling the elements of a $k$-subset modulo $n$. One then finds [1, Thm. 1.1(b)] that this triple $(X, X(q), C)$ exhibits the CSP. For example, taking $n = 4$ and $k = 2$, one has $c$ acting as shown here:

![Diagram](image)

One can compute $X(q) = 1 + q + 2q^2 + q^3 + q^4$ from (1). Note that $X(1) = 6$, while $X(e^{2\pi i}) = X(-1) = 2$ counts the two subsets $\{\{1,3\}, \{2,4\}\}$ fixed by $c^3$, and $X(e^{2\pi i/2}) = 0 = X((e^{2\pi i/3})^3)$ since no two-element subset is fixed by $c$ or $c^3$.

The CSP was first defined in [1]. It has proven to be remarkably ubiquitous -- see, for example, B. Sagan’s excellent survey [3]. The special case of a CSP when $C$ has order 2 was known as J. Stembridge’s $q = -1$ phenomenon [4]. He gave interesting examples involving enumeration of plane partitions and Young tableaux.

Stembridge emphasized the value of a single $q$-formula $X(q)$ encompassing both the cardinality of $X$ as $X(1)$, and a second enumeration $X(-1)$ of a symmetry class within $X$. A CSP triple $(X, X(q), C)$ generalizes his idea. The polynomial $X(q)$ packages as its $n^{th}$ root-of-unity evaluations, or equivalently in its residue class modulo $q^n - 1$, all of the information about the cyclic action of $C$ on $X$. In fact, given $(X, C)$ there is always a unique (but generally uninteresting) choice of a polynomial $X(q)$ of degree at most $n - 1$ completing the triple, as the CSP is

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equivalent [1, Prop. 2.1(ii)] to the assertion that
\[ X(q) = \sum_{i=0}^{n-1} a_i q^i \mod q^n - 1, \]
where \( a_i \) is the number of orbits of \( C \) on \( X \) in which the stabilizer cardinality divides \( i \). Thus a CSP interprets combinatorially the coefficients of \( X(q) \) when reduced \( \mod q^n - 1 \), e.g., \( a_0 \) counts the total number of orbits on \( X \), while \( a_1 \) counts the number of free orbits. Our proto-example with \( n = 4 \) and \( k = 2 \) has 
\[ X(q) \equiv 2 + q + 2q^2 + 3q^3 \mod q^4 - 1, \]
so \( a_0 = 2 \) counts the two orbits in total, and \( a_1 = 1 \) counts the free orbit.

Here is a second example from [1]. Let \( X \) be the set of triangulations of a regular \((n + 2)\)-gon, with \( C \) a cyclic group of order \( n + 2 \) rotating triangulations, and let
\[
X(q) = \frac{1}{[n+1]_q} \left[ \begin{array}{c} 2n \\ n \end{array} \right]_q,
\]
a \( q \)-Catalan number considered by P.A. MacMahon. Then \((X, X(q), C)\) exhibits the CSP [1, Thm. 7.1]. For example, when \( n = 4 \), the four orbits of triangulations are represented by
\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\end{array}
\]
while
\[
X(q) = \frac{1}{[5]_q} \left[ \begin{array}{c} 8 \\ 4 \end{array} \right]_q = 1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 2q^8 + q^9 + q^{10} + q^{12}
\equiv 4 + q + 3q^2 + 2q^3 + 3q^4 + q^5 \mod q^6 - 1,
\]
so that \( a_0 = 4 \) counts the four orbits, of which \( a_1 = 1 \) of them is free (the fourth orbit), while \( a_2 = 3 \) orbits (the first, second, fourth) have stabilizer size dividing \( 2 \), and \( a_3 = 2 \) orbits (the third, fourth) have stabilizer size dividing \( 3 \).

It was conjectured by the authors, and verified by S.-P. Eu and T.-S. Fu, that this triangulation example generalizes to a CSP triple \((X, X(q), C)\) in which \( X \) is the collection of clusters in a cluster algebra of finite type \( W \) à la S. Fomin and A. Zelevinsky, where \( C \) is generated by a deformed Coxeter element, and \( X(q) \) is a \( q \)-analogue of the Catalan number for \( W \).

So what makes a generating function \( X(q) \) “natural”? To some extent, this is in the eye of the beholder. Nevertheless, here are some conditions on \( X(q) \) arising in many CSP’s encountered so far:

(i) \( X(q) \) is the statistic generating function for a map \( s : X \to \{0, 1, 2, \ldots \} \), that is, \( X(q) = \sum_{x \in X} q^{s(x)} \).
(ii) \( X(q) \) has a simple product formula.
(iii) \( X(q^2) \) at \( q = p^d \) a prime power counts the points of a variety \( X(\mathbb{F}_q) \) defined over the finite field \( \mathbb{F}_q \).
(iv) \( X(q^2) = \sum_i \beta_i q^i \) records the Betti numbers \( \beta_i \) of a complex variety \( X(\mathbb{C}) \).
(v) \( X(q) = \sum_i \dim \mathcal{O}_i q^i \) records the Hilbert series of some interesting graded ring \( \mathcal{O} = \oplus_i \mathcal{O}_i \).
(vi) \( X(q^2) \) is, up to a power of \( q \), the formal character of an \( SL_2(\mathbb{C}) \)-representation, that is, the sum \( \sum_i \dim V_i q^i \) where \( V_i \) is the weight space on which a diagonal matrix with eigenvalues \((q, q^{-1})\) acts via the scalar \( q^i \).

Our proto-example has each of these natural properties.
(i) After multiplying \( X(q) \) by \( q^{\frac{k+1}{2}} \), it is the statistic generating function for \( k \)-subsets \( A \) by their sum \( s(A) = \sum_{a \in A} a \).

(ii) The product formula for \( X(q) \) is given in (1).

(iii) \( X(q^2) \) counts the points in the Grassmannian of \( k \)-planes in an \( n \)-dimensional vector space over \( F_q \).

(iv) \( X(q^2) \) records the Betti numbers for this Grassmannian over \( \mathbb{C} \).

(v) When the symmetric group \( S_n \) permutes polynomials in \( n \) variables, \( X(q) \) is the Hilbert series for the quotient ring of the polynomials invariant under \( S_k \times S_{n-k} \) after modding out the nonconstant polynomials invariant under \( S_n \).

(vi) \( q^{-\frac{k(n-k)}{2}} X(q^2) \) is the formal character for the \( k \)-th exterior power of the \( n \)-dimensional \( SL_2(\mathbb{C}) \)-irreducible.

In our triangulations example, the \( q \)-Catalan \( X(q) \) has an interpretation as in (i), (ii), (iii), and a variation of (v). We know no interpretation like (iv) or (vi).

Some CSP’s in the literature are proven via a linear algebra paradigm [1, §2]. Such proofs interpret \( X(q) \) as in (iv) or (v), giving a graded representation \( V = \bigoplus V_i \) of the cyclic group \( C \). One shows that \( X(\zeta^d) \) equals the size of the \( c^d \)-fixed subset of \( X \), by computing the trace of \( c^d \) using two bases. The first basis is indexed by \( C \) and permuted by \( c \), so that the trace of \( c^d \) is the size of the \( c^d \)-fixed subset. The second basis shows that \( c \) scales \( V_i \) by \( \zeta^i \), so that \( c \) has trace \( X(\zeta^d) \).

A pleasing situation where this paradigm works generalizes (v) above. It arises from the invariant theory of finite subgroups \( W \) of \( GL_n(\mathbb{C}) \) generated by reflections, that is, elements whose fixed space is a complex hyperplane. T. Springer developed a theory of regular elements in such groups, which are the elements \( c \) that have an eigenvector fixed by none of the reflections of \( W \). Using Springer’s main result, one obtains [1, Thm. 8.2] a CSP triple from the coset space \( X := W/W' \) for any subgroup \( W' \), with \( C \) generated by a regular element left-translating cosets, and \( X(q) \) is the quotient of the Hilbert series for the \( W' \)-invariant polynomials over the Hilbert series for the \( W \)-invariant polynomials.

An intriguing CSP was conjectured by D. White involving rectangular Young tableaux and the cyclic action of jeu-de-taquin promotion. It has now seen several proofs via the linear algebra paradigm, first by B. Rhoades [2], and most recently by B. Fontaine and J. Kamnitzer. Such insightful proofs are rarer than we would like. Many known instances of CSP’s, such as the triangulations example, have only been verified using a product formula for \( X(q) \) to evaluate \( X(\zeta^d) \), and comparing with known counts of symmetry classes.

We close with a perplexing example of this nature. Let \( X \) be the set of \( n \times n \) alternating sign matrices: the matrices with \( 0, \pm 1 \) entries whose row and column sums are all +1, and nonzero entries alternate in sign reading along any rows,
columns. Here they are for $n = 3$:

$$\begin{array}{cccc}
0 & 1 & 0 & c \\
1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0
\end{array} \quad \begin{array}{cccc}
0 & 1 & 0 & c \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0
\end{array} \quad \begin{array}{cccc}
0 & 0 & 1 & c \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1
\end{array}$$

Let $C$ be the cyclic group of order 4 whose generator $c$ rotates matrices through 90 degrees. Let

$$X(q) = \prod_{k=0}^{n-1} \frac{[3k + 1]!}{[n + k]! q}.$$ 

This triple $(X, X(q), C)$ exhibits the CSP, but we have no linear algebraic proof. Furthermore, $X(q)$ is only known as the generating function for descending plane partitions by weight, and is not defined by a statistic on alternating sign matrices.

**References**


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