FAKE GAUSSIAN SEQUENCES

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Abstract. Some positivity conjectures are made generalizing known results for Gaussian posets.

For a sequence of non-negative integers \( \bar{a} = (a_1, a_2, \cdots, a_n) \), and a non-negative integer \( m \), MacMahon [1, p. 137] considered the rational function of \( q \),

\[
F(\bar{a}, m, q) = \frac{\prod_{i=1}^{n} (1 - q^{m+1})^{a_i}}{\prod_{i=1}^{n} (1 - q^i)^{a_i}}.
\]

He was interested in the values of \( \bar{a} \) such that \( F(\bar{a}, m, q) \) is a polynomial in \( q \) for all non-negative integers \( m \). For example,

\[
F((1, 1, \cdots, 1), m, q) = \left[ \begin{array}{c} m + n \\ n \end{array} \right]_q,
\]

or if \( a_i \) is decreasing, \( F(\bar{a}, m, q) \) is a \( q \)-multinomial coefficient.

We say \( \bar{a} \) has the polynomial property if \( F(\bar{a}, m, q) \) is a polynomial in \( q \) for all non-negative integers \( m \). It is easy to restate this polynomiality condition as a condition on \( a = (a_1, a_2, \cdots, a_n) \).

Proposition 1. \( \bar{a} \) has the polynomial property if, and only if,

\[
\sum_i a_{ip} \leq \sum_i a_{ip+r}
\]

for all positive integers \( 1 \leq r < p \leq n \).

Proof. We just check the power of the cyclotomic polynomial \( \phi_p(q) \). In the denominator it appears \( \sum_i a_{ip} \) times, in the numerator \( \sum_i a_{ip+r} \) times, where \( m \equiv -r \) mod \( p \).

Proposition 1 with \( p = n \) clearly implies that \( a_r \geq a_n \), and since we may assume that \( a_n > 0 \), this implies each entry of \( \bar{a} \) is strictly positive. Also Proposition 1 says that the allowed set of \( \bar{a} \) is the set of integral points in a convex polyhedral cone \( C_n \) [4, §4.6].

One may ask if \( F(\bar{a}, m, q) \) has non-negative coefficients if \( \bar{a} \) has the polynomial property. If \( (a_1, a_2, \cdots, a_n) \) are the level numbers of a known Gaussian poset, this is known to be true [4, p. 270]. This we can think of any \( \bar{a} \) with the polynomial property as a fake Gaussian sequence. In general it is not true, for example

\[
F((1, 3, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1), 1, q) = 1 + q + \cdots - q^7 + \cdots,
\]
and it can be shown that \( \bar{a} = (1, 3, 1^{2s}, 2, 1^{2s+2}) \) also contains \(-q^7\) for any \( s \geq 3 \). I do not know of any such examples with fewer than 17 parts.

Nonetheless, there are some special families of \( \bar{a} \), which are not the level numbers of a known Gaussian poset, for which positivity does hold.

**Proposition 2.** If \( \bar{a} \) has at most 6 parts and the polynomial property, then \( F(\bar{a}, m, q) \) has non-negative coefficients.

*Proof.* The extreme rays of the cone \( C_6 \) are

\[
\{(1,0,0,0,0,0), (1,1,0,0,0,0), (1,2,1,0,0,0), (1,1,1,0,0,0), (1,1,1,1,0,0), \\
(1,2,2,1,0,0), (1,2,3,2,1,0), (1,3,3,2,1,0), (1,2,2,2,1,0), (1,3,2,1,1,0), \\
(1,1,2,1,1,0), (1,1,1,1,1,0), (1,2,1,1,1,0), (1,2,3,3,2,1), (1,2,2,2,2,1), \\
(1,1,1,1,1,1)\}. 
\]

All are known to be level numbers of Gaussian posets [4, p. 270], except

\[
\{(1,3,3,2,1,0), (1,2,1,1,1,0)\}. 
\]

There is simplicial triangulation of \( C_6 \) using these extreme rays and

\[
\{(1,2,2,1,1,0), (2,3,3,3,2,1)\}. 
\]

These facts have been verified using Porta. Thus any integer point in \( C_6 \) must be a positive integral combination of these 18 vectors. Since

\[
F(\bar{a} + \bar{b}, m, q) = F(\bar{a}, m, q)F(\bar{b}, m, q), 
\]

we must verify non-negativity at these remaining four points. A computer verification for a stronger statement using (1.3) is indicated below. \( \square \)

It is of interest to consider the symmetric case: \( a_i = a_{n+1-i} \) for all \( i \), this occurs for connected Gaussian posets. MacMahon [1, p. 141-144] listed the extreme rays for \( n \leq 8 \).

**Conjecture 1.** If \( \bar{a} \) is symmetric and has the polynomial property, then \( F(\bar{a}, m, q) \) has non-negative coefficients.

**Proposition 3.** Conjecture 1 holds for all \( \bar{a} \) with at most 10 parts.

*Proof.* We consider the cone \( \tilde{C}_n \), which is \( C_n \) with the additional equalities

\[
a_i = a_{n+1-i}, \quad 1 \leq i \leq n.
\]
The extreme rays of $\overline{C}_n$ for $n \leq 10$ are

\[
n = 1 \quad \{(1)\},
\]
\[
n = 2 \quad \{(1, 1)\},
\]
\[
n = 3 \quad \{(1, 1, 1), (1, 2, 1)\},
\]
\[
n = 4 \quad \{(1, 1, 1, 1), (1, 2, 2, 1), (1, 1, 1, 1, 1)\}
\]
\[
n = 5 \quad \{(1, 1, 2, 1, 1), (1, 2, 2, 2, 1), (1, 2, 3, 2, 1)\},
\]
\[
n = 6 \quad \{(1, 1, 1, 1, 1, 1), (1, 2, 2, 2, 2, 1), (1, 2, 3, 3, 2, 1)\},
\]
\[
n = 7 \quad \{(1, 1, 1, 1, 1, 1, 1), (1, 1, 1, 2, 1, 1, 1), (1, 2, 2, 2, 2, 1), (1, 2, 3, 3, 3, 2, 1), (1, 2, 3, 4, 3, 2, 1)\},
\]
\[
n = 8 \quad \{(1, 1, 1, 1, 1, 1, 1, 1), (1, 1, 2, 2, 2, 2, 1, 1), (1, 2, 2, 2, 2, 2, 2, 1),
\]
\[
(1, 2, 2, 3, 3, 2, 2, 1), (1, 2, 2, 3, 3, 3, 2, 1), (1, 2, 3, 3, 4, 3, 2, 1),
\]
\[
n = 9 \quad \{(1, 2, 3, 4, 5, 4, 3, 2, 1), (1, 2, 3, 3, 4, 3, 3, 2, 1), (1, 1, 2, 2, 3, 2, 2, 1, 1),
\]
\[
(1, 2, 2, 2, 3, 2, 2, 1, 1), (2, 2, 3, 4, 5, 4, 3, 2, 1), (1, 1, 1, 1, 2, 1, 1, 1, 1),
\]
\[
(1, 2, 3, 4, 4, 4, 3, 2, 1), (1, 2, 3, 3, 3, 3, 3, 2, 1), (1, 2, 2, 2, 2, 2, 2, 2, 1),
\]
\[
(1, 1, 1, 1, 1, 1, 1, 1, 1)\},
\]
\[
n = 10 \quad \{(1, 2, 3, 4, 5, 5, 4, 3, 2, 1), (2, 3, 5, 5, 6, 6, 5, 5, 3, 2), (2, 4, 4, 5, 6, 6, 5, 4, 4, 2),
\]
\[
(2, 3, 3, 4, 4, 3, 3, 2, 1), (1, 2, 3, 4, 4, 4, 3, 2, 1), (2, 3, 4, 6, 6, 6, 6, 4, 3, 2),
\]
\[
(1, 2, 2, 3, 3, 3, 2, 2, 1, 1), (2, 2, 3, 4, 4, 4, 3, 2, 2, 1), (1, 1, 2, 2, 2, 2, 2, 2, 1, 1),
\]
\[
(1, 2, 3, 3, 3, 3, 3, 2, 1), (1, 2, 2, 2, 2, 2, 2, 2, 2, 1), (1, 1, 1, 1, 1, 1, 1, 1, 1)\}\}
\]

while the possibly non-Gaussian extreme rays in these sets are

\[
\{(1, 1, 2, 2, 2, 2, 2, 1, 1), (1, 2, 3, 3, 3, 2, 2, 1), (1, 2, 3, 3, 3, 3, 3, 2, 1), (1, 2, 2, 2, 3, 2, 2, 2, 1),
\]
\[
(2, 2, 3, 4, 5, 4, 3, 2, 2), (2, 3, 5, 5, 6, 6, 5, 5, 3, 2), (2, 4, 4, 5, 6, 6, 5, 4, 4, 2),
\]
\[
(2, 3, 3, 4, 4, 3, 3, 3, 2), (2, 3, 4, 6, 6, 6, 6, 4, 3, 2), (1, 2, 2, 3, 3, 3, 2, 2, 1),
\]
\[
(2, 2, 3, 4, 4, 4, 3, 2, 2)\}\}
\]

For any $n \leq 10$ there is a simplicial triangulation of $\overline{C}_n$ using the appropriate extreme rays, thus we check only the possibly non-Gaussian extreme rays, using the technique below. $\square$

To prove a particular special case by computer we verify a stronger positivity result. It is clear that $F(\bar{a}, m, q)$ is a polynomial in $q^m$ of degree $a_1 + a_2 + \cdots + a_n$, and thus may be expanded

\[
F(\bar{a}, m, q) = \sum_{s=0}^{a_1+a_2+\cdots+a_n-1} \left[ \frac{m + a_1 + a_2 + \cdots + a_n - s}{a_1 + a_2 + \cdots + a_n} \right] q^s W_s(\bar{a}, q),
\]

for some rational function $W_s(\bar{a}, q)$ independent of $m$. Clearly, setting $m = 0$ in (1.2) implies

\[
W_0(\bar{a}, q) = 1
\]
and recursively

\[(1.3) \quad W_i(\bar{a}, q) = F(\bar{a}, i, q) - \sum_{s=0}^{i-1} W_s(\bar{a}, q) \left[ \frac{a_1 + a_2 + \cdots + a_n - s}{a_1 + a_2 + \cdots + a_n} \right]_q. \]

Clearly (1.3) implies that \(W_s(\bar{a}, q)\) is a polynomial in \(q\), for example

\[W_1(\bar{a}, q) = \prod_{j=1}^{n} \frac{(1 - q^{j+1})^{a_j}}{(1 - q^{j})^{a_j}} = \left[ \frac{a_1 + a_2 + \cdots + a_n + 1}{a_1 + a_2 + \cdots + a_n} \right]_q.\]

Also it is easy to see that \(W_s(\bar{a}, q) = 0\) for \(s > a_1 + a_2 + \cdots + a_n - n\), by setting \(m = -1, -2, \ldots, -(n - 1)\) in (1.2).

**Conjecture 2.** If \(\bar{a}\) is symmetric and has the polynomial property, then the polynomials \(W_s(\bar{a}, q)\) have non-negative coefficients.

It is clear that Conjecture 2 implies Conjecture 1. Conjecture 2 holds in the Gaussian poset case, when \(W_s(\bar{a}, q)\) is the generating function of linear extensions having \(s\) descents according to the major index. A special case of Conjecture 2 can be verified if \(m = 1\).

**Proposition 4.** Let \(c_i = a_i - a_{i-1}\), where \(a_0 = 0\). If \(c_1 \geq c_2 \geq \cdots \geq c_{\lfloor (n+1)/2 \rfloor}\), then Conjecture 2 holds for \(m = 1\).

**Proof.** Put \(p = \lfloor (n+1)/2 \rfloor\). We have

\[F(\bar{a}, 1, q) = \prod_{i=1}^{p} \left( 1 - \frac{q^{n+2-i}}{1 - q^i} \right)^{c_i} = \left[ \frac{n + 1}{p} \right]_q \prod_{i=1}^{p} \left[ \frac{n + 1}{p - i} \right]_q^{c_{p-i}}.\]

If \(c_i\) is decreasing, then each term is a \(q\)-binomial coefficient raised to a non-negative power. \(\square\)

**Proposition 5.** Conjecture 2 holds if \(\bar{a}\) has at most 10 parts.

**Proof.** The proof of [3, Prop. 12.6(ii)] implies

\[W_s(\bar{a} + \bar{b}, q) = \sum_{i,j} W_i(\bar{a}, q) W_j(\bar{b}, q) \left[ \frac{a_1 + a_2 + \cdots + a_n + j - i}{s - i} \right]_q \left[ \frac{b_1 + b_2 + \cdots + b_l + i - j}{s - j} \right]_q q^{(s-i)(s-j)},\]

so that the positivity of the coefficients of \(W_s(\bar{a})\) is preserved under sum. Again we check \(W_s(\bar{a})\) for the vectors \(\bar{a}\) given in Proposition 3. This was done explicitly on a computer using (1.3). \(\square\)

Proposition 2 was also verified, by checking the non-negativity of \(W_s(\bar{a})\) for the vectors \(\bar{a}\) listed there.

It is not hard to verify that

\[W_s(\bar{a}, 1/q) q^{s(a_1 + a_2 + \cdots + a_n)} = W_s(\bar{a}, q).\]
If the coefficients of $W_s(\bar{a})$ are “centered” and unimodal, then (1.2) implies that the coefficients of $F(\bar{a}, m, q)$ are also unimodal. However the coefficients of $W_s(\bar{a})$ and $F(\bar{a}, m, q)$ are not always unimodal, e.g. $\bar{a} = (1, 2, 2, 3, 3, 3, 3, 2, 2, 1)$, $m = 1$, and $s = 1$.

If $\bar{a}$ does arise from a Gaussian poset, then the coefficient of $q^j$ in $F(\bar{a}, m, q)$ is an increasing function of $m \geq 0$. This property appears also to hold for $\bar{a}$ with the polynomial property.

**Conjecture 3.** If $\bar{a}$ has the polynomial property, then the coefficient of $q^j$ in $F(\bar{a}, m, q)$ is an increasing function of $m \geq 0$.

If $W_s(\bar{a})$ has non-negative coefficients, then (1.2) verifies Conjecture 3. So the validity Conjecture 2 implies the validity Conjecture 3.

The $m \to \infty$ limit of (1.2) can be rewritten

$$g(\bar{a}, q) = \frac{\prod_{i=1}^{\alpha_1+\alpha_2+\cdots+\alpha_n}(1-q^i)}{\prod_{i=1}^{n}(1-q^i)^{\alpha_i}} = \sum_{s=0}^{\alpha_1+\alpha_2+\cdots+\alpha_n-n} W_s(\bar{a}, q).$$

**Conjecture 4.** If $a_i > 0$ for all $i$ and $g(\bar{a}, q)$ is a polynomial, then the coefficients of $g(\bar{a}, q)$ are non-negative.

If $F(\bar{a}, m, q)$ is a polynomial for all non-negative $m$, then $g(\bar{a}, q)$ is also a polynomial. So Conjecture 2 implies Conjecture 4 in these cases. However, $\bar{a}$ may not have the polynomial property, yet $g(\bar{a}, q)$ is a polynomial (e.g. $\bar{a} = (1, 3, 2)$).

If $\bar{a}$ is decreasing, then $g(\bar{a}, q)$ is a $q$-multinomial coefficient. Conjecture 4 asks for an interpretation of $g(\bar{a}, q)$ as the generating function for a statistic on some set of multiset permutations. Similarly, Conjecture 2 asks for an interpretation of $W_s(\bar{a})$ as the generating function of some set of permutations with $s$ descents.

**Remarks**

1. It is easy to see that if $F(\bar{a}, m, q)$ is polynomial for all non-negative $m$, then it is also a Laurent polynomial for all non-positive $m$.

2. $W_s(\bar{a})$ can have negative coefficients even though $F(\bar{a}, m, q)$ does not: $\bar{a} = (1, 3, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, $m = 1$, and $s = 1$.

3. The number of new extreme rays for $C_n$ reported by Porta is given below.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td># of new extreme rays for $n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>7</td>
<td>3</td>
<td>19</td>
<td>18</td>
<td>40</td>
<td>99</td>
</tr>
</tbody>
</table>

4. The number of extreme rays for $n$ in the symmetric version of Conjecture 1 is given below.

<table>
<thead>
<tr>
<th>$n$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td># of extreme symmetric rays for $n$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>6</td>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>

5. For the 93 extreme rays $\bar{a}$ in $C_9$, $W_s(\bar{a})$ has non-negative coefficients.

6. There is no connected Gaussian poset with level numbers $(1, 1, 2, 2, 2, 2, 1, 1)$ or $(1, 2, 2, 3, 3, 2, 2, 1)$, these are the first candidates for a new Gaussian poset.
(7) An explicit version of (1.2) is

\[ F((1, 1, 2, 2, 2, 1, 1), q, m) = \binom{m+12}{12}_q +\]
\[ (q^3 + q^4 + q^5 + 2q^6 + q^7 + q^8 + q^9) \binom{m+11}{12}_q +\]
\[ (q^8 + q^9 + 2q^{10} + 2q^{11} + 3q^{12} + 2q^{13} + 2q^{14} + q^{15} + q^{16}) \binom{m+10}{12}_q +\]
\[ (q^{15} + q^{16} + q^{17} + 2q^{18} + q^{19} + q^{20} + q^{21}) \binom{m+9}{12}_q + \binom{m+8}{12}_q. \]

Is this the character of a natural \( sl_2 \) representation, as occurs for the known Gaussian posets [2]?

(8) Conjecture 1 has been verified for \( \bar{\alpha} = (\text{Reverse}(\lambda), \lambda) \), of length \( n \leq 20 \), and partitions \( \lambda, |\lambda| \leq 20 \).

(9) Conjecture 2 has been verified for \( a_1 + a_2 + \cdots + a_n \leq 15 \)

(10) Conjecture 4 has been verified for \( a_1 + a_2 + \cdots + a_n \leq 18 \).

References


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