Ramanujan Continued Fractions Via Orthogonal Polynomials

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Abstract

Some Ramanujan continued fractions are evaluated using asymptotics of polynomials orthogonal with respect to measures with absolutely continuous components.

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1 Introduction

A sequence of orthogonal polynomials satisfies a three-term recurrence relation

\[ y_{n+1}(x) = (x - b_n)y_n(x) - \lambda_n y_{n-1}(x), \quad n \geq 0, \]

with \( \lambda_{n+1} > 0 \) and \( b_n \in \mathbb{R} \) for \( n = 0, 1, \ldots \). There are two special independent solutions to (1.1), denoted \( p_n \) and \( p_n^* \), with the initial conditions \( p_0 = 1, p_{-1} = 0, p_0^* = 0, p_1^* = 1 \). The finite continued fraction associated with (1.1) is

\[
C_n(x) = \frac{1}{x - b_0 - \frac{\lambda_1}{x - b_1 - \frac{\lambda_2}{x - b_2 - \cdots - \frac{\lambda_{n-1}}{x - b_{n-1}}}}} = \frac{p_n^*(x)}{p_n(x)}.
\]

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The above $C_n(x)$ is the $n$th convergent of the continued fraction
\[
\frac{1}{x - b_0 - \frac{\lambda_1}{x - b_1 - \frac{\lambda_2}{\ddots}}}.
\]

$n = 1, 2, \ldots$. Continued fractions of the above type are called $J$-fractions [12] and are associated with orthogonal polynomials.

It is clear that knowing the large $n$ asymptotics of $p_n(x)$ and $p'_n(x)$ enables us to evaluate the infinite continued fraction $C_\infty(x)$ via
\[
C_\infty(x) = \lim_{n \to \infty} C_n(x).
\]

This has been done for a generalization of the Rogers-Ramanujan continued fraction in [1].

Andrews et al. [4] (see our Theorem 4.7) have recently evaluated another Ramanujan continued fraction $C_n(1/2)$, whose limit depends upon the congruence class of $n$ modulo 3. In this paper we reprove the evaluation via asymptotics of orthogonal polynomials, and explain that the modulo 3 behavior is typical for continued $J$-fractions evaluated at $x = \cos(\pi/3)$ when the corresponding orthogonal polynomials are orthogonal with respect to a weight function supported on $[-1, 1]$. As a byproduct we evaluate several new continued fractions, see Theorems 4.1-4.6. The modulo 3 behavior can be generalized to any modulus $k$, by choosing a suitable value for $x$ which depends upon $k$. For the Ramanujan continued fraction, this result is given in Theorem 4.8. Moreover the modulo $k$ behavior holds for a wide class of orthogonal polynomials which have the appropriate asymptotics. We discuss this in §5, and give two such general theorems for modulo $k$ limits, Theorems 5.2 and 5.6. In §6 we demonstrate these ideas using the Al-Salam-Chihara and $q$-ultraspherical polynomials. After completing this manuscript the work of Bowman and McLaughlin [7] was brought to our attention. In §7 we will compare their results with our own.

2 A family of $q$-Chebyshev polynomials

In this section we collect the basic facts about a system of orthogonal polynomials introduced by Ismail and Mulla in [11]. We state the defining recurrence relation, give generating functions and asymptotics. The polynomial system considered is a $q$ extension of the Chebyshev polynomials. These polynomials will be used to explain and extend the results by Andrews et al.

Consider the three-term recurrence relation
\[
y_{n+1}(x) = (2x + aq^n)y_n(x) - y_{n-1}(x).
\]

We define the polynomial system $\{p_n(x; a, q)\}$ to be the solution of (2.1) with the initial conditions
\[
p_0(x; a, q) = 1, \quad p_1(x; a, q) = 2x + a.
\]
The polynomials of the second kind $p_n^*(x; a, q)$ are defined as the solutions to (2.1) with the initial conditions

$$p_n^0(x; a, q) = 0, \quad p_n^1(x; a, q) = 2.$$

It readily follows that

$$p_n^*(x; a, q) = 2p_{n-1}(x; a, q).$$

The generating function of $\{p_n(x; a, q)\}$ is easily found from (2.1) to be

$$P(x, t) = \sum_{n=0}^{\infty} p_n(x; a, q)t^n = \sum_{k=0}^{\infty} \frac{a^k t^k q^j(0)}{(t e^{i\theta}, t e^{-i\theta}; q)_{k+1}}, \quad x = \cos \theta,$$

see [11].

One can apply Darboux’s method to the generating function (2.2) and find the leading term in the large $n$ asymptotics of $p_n(x; a, q)$. Let

$$F(z, a, q) := \sum_{n=0}^{\infty} \frac{a^n z^n q^n(0)}{(q, q / z^n; q)_n}.$$

We parameterize $x$ by $x = \cos \theta$, so that $e^{\pm i\theta} = x \pm \sqrt{x^2 - 1}$. The branch of the square root is chosen such that $\sqrt{x^2 - 1}/x \to 1$ as $x \to \infty$. We shall use the notation

$$\rho(x), 1/\rho(x) = x \pm \sqrt{x^2 - 1}, \quad |\rho(x)| > 1 \text{ for } x \notin [-1, 1].$$

**Proposition 2.1.** If $x \notin [-1, 1]$, then the asymptotic relation

(A) \hspace{1cm} p_n(x; a, q) = \frac{[\rho(x)]^{n+1}}{\rho(x) - 1/\rho(x)} F(z, a, q)(1 + o(1)), \quad \text{as } n \to \infty,

holds. On the other hand if $x \in [-1, 1]$, and $z = e^{i\theta}$ then

(B) \hspace{1cm} p_n(x; a, q) = \frac{z^n}{1 - z^2} F(z, a, q) + \frac{z^{-n}}{1 - z^2} F(1/z, a, q) + o(1), \quad \text{as } n \to \infty.

**Proposition 2.2.** The asymptotic function $F(z, a, q)$ has the alternative representation

$$F(z, a, q) = (-a/z; q)_{\infty} \sum_{k=0}^{\infty} \frac{(aq z^{-3})^k q^{3k}}{(q, q / z^n, -a/z; q)_k}.$$

**Proof.** Apply the $\phi_1$ to $\phi_2$ transformation

$$\phi_1(a, b; c; q, z) = \frac{(az; q)_{\infty}}{(z; q)_{\infty}} \phi_2(b, c; a; q, bz),$$

[9, (IV.4)], with $a = q^{-n}, \quad b = q^n, \quad c = q z^{-2}, \quad z = -aq^n/z,$ and let $n \to \infty$. \hfill \Box

We choose $b_n = -aq^n$, $\lambda_n = 1$, and replace $x$ by $2x$ in $C_n(x)$. In view of Proposition 2.1, to evaluate $C_n(x)$, one must choose $a$ and $z$ so that $F(a, z, q)$ is evaluable. We collect these choices in the next section.
3 Evaluations of the asymptotic function

In this section we evaluate the asymptotic function \( F(z, a, q) \) for various choices of \( z \) and \( a \).

**Proposition 3.1.** If \( \omega = e^{2\pi i/3} \), we have

(A) \[ F(e^{i\pi/3}, q, q) = (\omega q; q)_\infty (q^2; q^3)_\infty \]

(B) \[ F(e^{i\pi/3}, 1, q) = (\omega q; q)_\infty \left[(q; q^3)_\infty - \omega(q^2; q^3)_\infty\right] \]

(C) \[ F(e^{i\pi/3}, 1/4, q) = \frac{\omega}{q} (\omega q; q)_\infty \left[(1 + q)(q; q^3)_\infty + \omega(q^2; q^3)_\infty\right] \]

(D) \[ F(e^{i\pi/3}, q^2, q) = (\omega q; q)_\infty \left[(1 + \omega^2)(q; q^3)_\infty - \omega^2(q^2; q^3)_\infty\right] \]

**Proof.** For the first evaluation, put \( z = e^{i\pi/3} \) in Proposition 2.2 and use the \( q^3 \)-binomial theorem. For (B), note that by Proposition 2.2, if \( \omega = e^{2\pi i/3} \)

\[ F(e^{i\pi/3}, 1, q) = \frac{(\omega; q)_\infty}{1 - \omega} \sum_{k=0}^{\infty} \left( \frac{q^{k+3}(q^3)}{(q, q^2, q \omega; q)_k} \right)^k (1 - \omega q^k) \]

Then use the \( q^3 \)-binomial theorem twice to obtain the stated result. The proofs of (C) and (D) are analogous to that of (B) and will be omitted. \( \Box \)

The evaluation in (A) was first proved by Andrews in [2].

**Proposition 3.2.** The following Rogers-Ramanujan identities

\[
F(p^{-1/2}, p^{3/2}, p^2) = (1 - p) \sum_{n=0}^{\infty} \frac{p^{n^2 + n}}{(p; p)_{2n+1}} \\
= (p^3, p^7, p^{10}, p^{10})_\infty (p^3, p^{16}, p^{20})_\infty \left/(p^2; p)_\infty\right.
\]

(3.1)

\[
F(p^{-1/2}, p^{5/2}, p^2) = (1 - p) \sum_{n=0}^{\infty} \frac{p^{n^2 + 2n}}{(p; p)_{2n+1}} \\
= (p^3, p^7, p^{10}, p^{10})_\infty (p^2, p^{18}, p^{20})_\infty \left/(p^2; p)_\infty\right.
\]

(3.2)

hold for \( 0 < p < 1 \).

**Proof.** These are equations (94) and (96) on Slater’s list of Rogers-Ramanujan type identities [16]. \( \Box \)

The results in Proposition 3.2 are valid for complex \( p \) with \( |p| < 1 \). The square roots present no problem because Proposition 2.2 indicates that the left-hand sides of (3.1) and (3.2) are analytic functions of \( p \) for \( |p| < 1 \). We shall need the evaluations that correspond to shifting \( a \) by \( q \) in Proposition 2.4.
Proposition 3.3. The following evaluations of the function $F$ hold

\[
F \left( p^{-1/2}, p^{7/2}, p^2 \right) = (1 - p) \sum_{n=0}^{\infty} \frac{p^{2n+2}}{p^{2n+1}}
\]

(3.3)

\[
= \frac{1}{(p; p)_{2+1}} \left[ - (p^1, p^9, p^{10})_\infty (p^8, p^{12}, p^{20})_\infty + (p^3, p^7, p^{10})_\infty (p^4, p^{16}, p^{20})_\infty \right],
\]

\[
F \left( p^{-1/2}, p^{6/2}, p^2 \right) = (1 - p) \sum_{n=0}^{\infty} \frac{p^{2n+4}}{p^{2n+3}}
\]

(3.4)

\[
= \frac{1}{(p; p)_{3+1}} \left[ (p^2, p^8, p^{10})_\infty (p^6, p^{14}, p^{20})_\infty + (p - 1)(p^4, p^6, p^{10})_\infty (p^2, p^{18}, p^{20})_\infty \right].
\]

Proof. Observe that

\[
F(\alpha, p^2) = (1 - p) \sum_{k=0}^{\infty} \frac{p^{k+2}}{(p; p)_{2k+1}}
\]

\[
= p^{-1}(1 - p) \sum_{k=0}^{\infty} \frac{p^{k+2k+1}}{(p; p)_{2k+1}}
\]

\[
= p^{-1}(1 - p) \sum_{k=0}^{\infty} \frac{p^{k+2}}{(p; p)_{2k}}
\]

Thus if $\alpha = p$, these two sums may be evaluated by Slater (99) and (94) respectively. This is the first result. For $\alpha = p^2$, the second sum may be evaluated using formula (96) of [16], while the first sum is

\[
\sum_{k=0}^{\infty} \frac{p^{k+2}}{(p; p)_{2k}} = \sum_{k=0}^{\infty} \frac{p^{2k+1}}{(p; p)_{2k+1}} = \sum_{k=0}^{\infty} \frac{p^{k+1}^2}{(p; p)_{2k+1}} + \sum_{k=0}^{\infty} \frac{p^{k+2}}{(p; p)_{2k}},
\]

again evaluable from (98) and (96) of Slater [16]. This establishes the second part and the proof is complete. 

Analogous to Proposition 3.3, we state without proof three other evaluations which also follow from Slater, (94), (96), (98), and (99).

Proposition 3.4. We have

\[
F \left( p^{-3/2}, p^{1/2}, p^2 \right) = (1 - p)(1 - p^3) \sum_{n=0}^{\infty} \frac{p^{n+2}}{(p; p)_{2n+1}(1 - pq^{2n+3})}
\]

(3.5)

\[
= \frac{(1 - p^3)}{(p^2; p)_{\infty}} \left[ (p^1, p^9, p^{10})_\infty (p^8, p^{12}, p^{20})_\infty + p(p^3, p^7, p^{10})_\infty (p^4, p^{16}, p^{20})_\infty \right].
\]
\[ F(p^{-3/2}, p^{3/2}, p^2) = (1 - p)(1 - p^3) \sum_{n=0}^{\infty} \frac{p^{n^2+3n}}{(p; p)_{2n+1}(1 - p^{2n+3})} \]

(3.6)

\[ = \frac{(1 - p^3)}{p^3(p^2; p)_\infty} \left[ (p^1, p^9, p^{10}, p^{10}) \pdq \left( p^8, p^{12}, p^{20} \right)_\infty \\
+ (p - 1)(p^3, p^7, p^{10}, p^{10}) \pdq \left( p^4, p^{16}, p^{20} \right)_\infty \right], \]

\[ F(p^{-3/2}, p^{3/2}, p^2) = (1 - p)(1 - p^3) \sum_{n=0}^{\infty} \frac{p^{n^2+5n}}{(p; p)_{2n+1}(1 - p^{2n+3})} \]

(3.7)

\[ = \frac{(1 - p^3)}{p^3(p^2; p)_\infty} \left[ (p^2 + 1)(p^1, p^9, p^{10}, p^{10}) \pdq \left( p^8, p^{12}, p^{20} \right)_\infty \\
+ (-p^2 + p - 1)(p^3, p^7, p^{10}, p^{10}) \pdq \left( p^4, p^{16}, p^{20} \right)_\infty \right]. \]

Slater’s (94), (96), (98) and (99) give the next set of evaluations.

**Proposition 3.5.** We have

(A) \[ F(i, iq, q^{2}) + F(-i, iq, q^{2}) = 2 \left( \frac{q^8, q^{32}, q^{40}, q^{40}}{q^{4}, q^{40}} \right)_\infty \left( \frac{q^{24}, q^{56}, q^{80}}{q^{24}, q^{56}, q^{80}} \right)_\infty, \]

(B) \[ F(i, iq^3, q^{2}) + F(-i, iq^3, q^{2}) = 2 \left( \frac{q^4, q^{36}, q^{40}, q^{40}}{q^{4}, q^{36}} \right)_\infty \left( \frac{q^{32}, q^{48}, q^{80}}{q^{32}, q^{48}, q^{80}} \right)_\infty, \]

(C) \[ F(i, iq, q^{2}) - F(-i, iq, q^{2}) = 2 q^{4} \left( \frac{q^{12}, q^{28}, q^{40}, q^{40}}{q^{4}, q^{36}} \right)_\infty \left( \frac{q^{16}, q^{64}, q^{80}}{q^{16}, q^{64}, q^{80}} \right)_\infty, \]

(D) \[ F(i, iq^3, q^{2}) - F(-i, iq^3, q^{2}) = 2 q^{3} \left( \frac{q^{16}, q^{24}, q^{40}, q^{40}}{q^{4}, q^{36}} \right)_\infty \left( \frac{q^{8}, q^{72}, q^{80}}{q^{8}, q^{72}, q^{80}} \right)_\infty. \]

Propositions 3.1–3.5 give examples of evaluations of \( F \) at special points of \( F(z, a, q) \) and \( F(z, aq, q) \). We can also find recurrences in \( a \), which show that if we can evaluate \( F(z, a, q) \) and \( F(z, aq, q) \), then we can find \( F(z, aq^n, q) \) for any integer \( n \).

**Proposition 3.6.** Let \( G_n = F(z, aq^n, q)/z^n \). Then \( G_n \) has the property

\[ G_{n+1} = -p_{n-1}(x; aq, q)G_0 + p_n(x; a, q)G_1, \]

or equivalently,

(3.8)

\[ F(z, aq^{n+1}, q) = -p_{n-1}(x; aq, q)z^{n+1}F(z, a, q) \\
+ p_n(x; a, q)z^nF(z, aq, q). \]

**Proof.** A routine calculation shows that

(3.9) \[ F(z, aq^{n+1}, q) = (1 + z^2 + aq^{n-1}z)F(z, aq^n, q) - z^2F(z, aq^{n-1}, q) \]

which implies that \( \{G_n\} \) satisfies the recurrence

\[ y_{n+1} = (2x + aq^{n-1})y_n - y_{n-1}. \]
Since \( \{ p_n(x; a, q), p_{n-1}(x; a, q) \} \) is a basis of solutions of the above recurrence, \( G_n \) must be a linear combination of \( p_n(x; a, q) \) and \( p_{n-1}(x; a, q) \) and the proposition follows.

Note that the way \( \{ p_n \} \) appears in (3.9) is similar to the way Lommel polynomials arise from Bessel functions, see [19].

4 Continued fractions

Let

\[
C_n(x; a, q) = \frac{p_n(x; a, q)}{2p_n(x; a, q)} = \frac{1}{2x + a -} \frac{1}{2x + aq -} \frac{1}{2x + aq^2 -} \cdots - \frac{1}{2x + aq^{n-1}}
\]

In this section we apply the asymptotic evaluations of §3 to evaluate

\[
C_\infty(x; a, q) = \lim_{n \to \infty} C_n(x; a, q).
\]

When there is only one term in the asymptotic expansion of Proposition 2.1, the \( n \) dependence is simple, and \( C_\infty(x; a, q) \) exists. If \( x \) is chosen so that two terms exist in Proposition 2.1, \( C_\infty(x; a, q) \) may not exist. However, restricting \( n \) to a congruence class, when \( x \) is suitably chosen, will yield a limit.

First we consider the single term case. As before we set \( x = \cos \theta, \ z = e^{i\theta} \), and \( |e^{-i\theta}| < |e^{i\theta}| \), and it follows from Proposition 2.1 that

\[
(4.1) \quad C_\infty(x; a, q) = \lim_{n \to \infty} \frac{p_n(x; a, q)}{2p_n(x; a, q)} = \frac{F(z, aq, q)}{zF(z, a, q)}.
\]

Moreover

\[
C_\infty(x; aq^m, q) = \lim_{n \to \infty} \frac{p_n(x; aq^m, q)}{2p_n(x; aq^m, q)} = \frac{F(z, aq^{m+1}, q)}{zF(z, aq^m, q)},
\]

hence

\[
C_\infty(x; aq^m, q) = \frac{-p_{m-1}(x; aq, q)z^{m+1}F(z, a, q) + p_m(x; a, q)z^mF(z, aq, q)}{-p_m(x; a, q)z^{m+1}F(z, a, q) + p_{m-1}(x; a, q)z^mF(z, aq, q)}.
\]

holds for \( x \notin [-1, 1] \).

For our first theorem we put \( q = p^2, \ 2x = p^{1/2} + p^{-1/2}, \ a = p^{1/2}, \ \) and use Propositions 3.2 and 3.3 to evaluate \( F(p^{-1/2}, p^{3/2}, p^2) \) and \( F(p^{-1/2}, p^{5/2}, p^2) \).
Theorem 4.1. If $0 < p < 1$, then

$$1 + p + p^2 - \frac{1}{1 + p + p^2 - \frac{1}{\ddots}} = 1 + p + p^{2^n} - \frac{1}{1 + p + p^{2^n} - \frac{1}{\ddots}} = \frac{1}{p} - \frac{1}{p} \frac{(p, p^3, p^{10})_\infty (p^8, p^{12}, p^{20})_\infty}{(p^3, p^6, p^{10})_\infty (p^9, p^{18}, p^{20})_\infty}.$$ 

Theorem 4.2 below follows from the parameter identifications $q = p^2$, $2x = p^{1/2} + p^{-1/2}$, $a = p^{5/2}$, and the use of Propositions 3.2 and 3.3 to evaluate $F(p^{1/2}, p^{5/2}, p^2)$ and $F(p^{-1/2}, p^{3/2}, p^2)$.

Theorem 4.2. If $0 < p < 1$, then

$$1 + p + p^2 - \frac{1}{1 + p + p^2 - \frac{1}{\ddots}} = 1 + p + p^{2^n} - \frac{1}{1 + p + p^{2^n} - \frac{1}{\ddots}} = \frac{1}{p} - \frac{1}{p} \frac{(p^2, p^8, p^{10})_\infty (p^6, p^{14}, p^{20})_\infty}{(p^3, p^6, p^{10})_\infty (p^9, p^{18}, p^{20})_\infty}.$$ 

To prove Theorem 4.3 we set $q = p^2$, $2x = p^{3/2} + p^{-3/2}$, $a = p^{1/2}$, and apply Proposition 3.4 to evaluate $F(p^{-3/2}, p^{3/2}, p^2)$ and $F(p^{-3/2}, p^{5/2}, p^2)$.

Theorem 4.3. If $0 < p < 1$, then

$$1 + p^3 + p^2 - \frac{1}{1 + p^3 + p^2 - \frac{1}{\ddots}} = 1 + p^3 + p^{2n} - \frac{1}{1 + p^3 + p^{2n} - \frac{1}{\ddots}} = \frac{1}{p^3} - \frac{1}{p^3} \frac{(p^3, p^7, p^{10})_\infty (p^4, p^{16}, p^{20})_\infty}{(p^3, p^7, p^{10})_\infty (p^4, p^{16}, p^{20})_\infty}.$$ 

For Theorem 4.4 we put $q = p^2$, $2x = p^{3/2} + p^{-3/2}$, $a = p^{5/2}$, and use Proposition 3.4 to evaluate $F(p^{-3/2}, p^{5/2}, p^2)$ and $F(p^{-3/2}, p^{9/2}, p^2)$.
Theorem 4.4. If $0 < p < 1$, then
\[
1 + p^3 + p^4 - \frac{p^7}{1 + p^3 + p^6 - \frac{p^3}{1 + p^3 + p^8 - \frac{p^3}{1 + p^3 + p^{2n+2} - \frac{p^3}{\ldots}}}}
\]
\[
= \frac{(p^2 + 1)(p, p^9, p^{10})_\infty (p^8, p^{12}, p^{20})_\infty - (p^2 - p + 1)(p^3, p^7, p^{10})_\infty (p^4, p^{16}, p^{20})_\infty}{p^3 ((p, p^9, p^{10})_\infty (p^8, p^{12}, p^{20})_\infty + (p - 1)(p^3, p^7, p^{10})_\infty (p^4, p^{16}, p^{20})_\infty}.
\]
Next we turn to evaluations which correspond to two terms in the asymptotic formula of Proposition 2.1. The first example has $x = 0$, $z = i$, so that
\[
2p_n(0; a, q^2) \approx i^n F(i, a, q^2) + i^{-n} F(-i, a, q^2).
\]
There are two possible limits, for $n$ even and $n$ odd. Proposition 3.5 evaluates the sums and differences at $a = iq$ and $a = iq^3$, implying the next two theorems.

Theorem 4.5. If $0 < p < 1$, then
\[
\lim_{n \to \infty} \frac{1}{p} - \frac{1}{1 - p^3} - \frac{1}{1 - p^5} - \cdots = \frac{p^3 (p^{16}, p^{64}; q^{80})_\infty}{(p^{32}, p^{64}, p^{80})_\infty}
\]

Theorem 4.6. If $0 < p < 1$, then
\[
\lim_{n \to \infty} \frac{1}{p} - \frac{1}{1 - p^3} - \frac{1}{1 - p^5} - \cdots = \frac{(p^4, p^{36}, p^{64}; q^{80})_\infty (p^{32}, p^{64}, q^{80})_\infty}{p(p^{16}, p^{64}, q^{80})_\infty (p^{16}, p^{64}, q^{80})_\infty}
\]

The evaluation of Andrews et al [4] chooses $z = e^{i\pi/3}$, $2x = 2 \cos(\pi/3) = 1$, and $a = 1$ and uses Proposition 3.1. In this case Proposition 2.1 contains two terms for the asymptotics of $p_n(x; a, q)$, and one must restrict $n$ to congruence classes modulo 3 to get a limiting result. For example, if $\epsilon = 0, 1$, or $-1$, and $z = e^{i\pi/3}$, then
\[
\lim_{N \to \infty} C_{3N + \epsilon + 1}(1/2; 1, q) = \lim_{N \to \infty} \frac{p^5_{3N + \epsilon + 1}(1/2; 1, q)}{2p^3_{3N + \epsilon + 1}(1/2; 1, q)}.
\]
Therefore
\[
\lim_{N \to \infty} C_{3N+\epsilon+1}(1/2; 1, q) = \frac{z^{1+\epsilon} F(e^{i\pi/3}, q, q) - z^{-1-\epsilon} F(e^{-i\pi/3}, q, q)}{z^{2+\epsilon} F(e^{i\pi/3}, 1, q) - z^{-2-\epsilon} F(e^{-i\pi/3}, 1, q)}
\]  
(4.3)  

In view of Proposition 3.1, we have evaluated the continued fraction “\(C_\infty(1/2; 1, q)\)” with three different limits. This is equivalent to Theorem 2.8 in [4], which is stated below.

**Theorem 4.7.** Let \(\epsilon = 0, 1\) or \(-1\).

\[
\lim_{N \to \infty} \frac{1}{1 + q - \frac{1}{1 + q^2 - \frac{1}{\ddots - \frac{1}{1 + q^{3N+\epsilon}}}}} = -\omega^2 (q^2; q^3)_\infty (q^{2\omega}; q)_\infty \omega^{k+1} (q\omega; q)_\infty
\]

\[
(\frac{q; q^2}{q, q^3})_\infty (\frac{q\omega^2; q^3}{q^2, q^4})_\infty - \omega^{k-1} (q\omega; q)_\infty
\]

In fact it is straightforward to prove the following generalization of (4.3), or Theorem 4.7, to the case of \(k\) different limits.

**Theorem 4.8.** If \(0 \leq s \leq k - 1\), and \(\omega := e^{2\pi i/k}\) then

\[
\lim_{N \to \infty} C_{Nk+s}(\cos(j\pi/k); a, q) = \frac{e^{ij\pi/k} \omega^{j\pi/k} F(e^{i\pi/k}, aq, q) - F(e^{-i\pi/k}, aq, q)}{\omega^{j\pi/k} F(e^{i\pi/k}, a, q) - F(e^{-i\pi/k}, a, q)}
\]

holds for \(1 \leq j < k\).

5 **Generalizations**

Assume that \(\{p_n(x)\}\) is generated by

\[
p_0(x) := 1, \quad p_1(x) = (x - b_0)/a_1, \quad x_{p_n}(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad n > 0,
\]

where \(b_n \in \mathbb{R}\) and \(a_n > 0\) for all \(n\). The spectral theorem for orthogonal polynomials asserts that there is a probability measure \(\mu\) such that \(\{p_n(x)\}\) is orthonormal with respect to \(\mu\).

**Theorem 5.1.** ([14, Theorem 40, p. 143]) Assume that

\[
\sum_{n=1}^{\infty} \{ |b_n| + |a_n - 1/2| \} \text{ converges.}
\]
Then the orthogonality measure \( \mu \) has the decomposition \( d\mu = \mu'(x)dx + dv \) where \( \mu' \) is continuous and positive on \((-1,1)\) and its support is \([-1,1]\). Moreover the function \( \nu \) is a jump function (a step function with possibly infinitely many jumps), which is constant on \((-1,1)\). Furthermore

\[
(5.3) \quad \lim_{n \to \infty} \left\{ \sin \theta p_n(\cos \theta) - \sqrt{\frac{2}{\pi}} \frac{\sin \theta}{\mu'(\cos \theta)} \sin[(n+1)\theta - \phi(\theta)] \right\} = 0,
\]

uniformly on every compact subinterval of \((0,\pi)\). Here \( \phi \) may depend on \( \theta \) but not on \( n \).

The angle \( \phi = \phi(\theta) \) is called the phase shift. For more information on asymptotics of general orthogonal polynomials the interested reader may consult Nevai’s survey article [15]. It is worth mentioning that the polynomials of §2 correspond to the case \( a_n = 1/2 \) and \( b_n = aq^n/2 \).

Assume that \( \{p_n(x)\} \) satisfies (5.1) and the condition (5.2). Let \( \{p_n^*(x)\} \) be a solution to the recursion in (5.1) with the initial conditions

\[
p_n^*(x) := 0, \quad p_1^*(x) := 1/a_1.
\]

The polynomials \( \{p_{n+1}(x) : n = 0, \ldots\} \) are orthogonal polynomials and \( p_{n+1}(x) \) is of exact degree \( n \). If the recursion coefficients in (5.1) satisfy the condition (5.2) then \( \{a_1p_{n+1}(x)\} \) is also a system of orthonormal polynomials and (5.3) holds for its measure of orthogonality.

Theorem 5.2. Assume that the convergence condition (5.2) holds and that \( \mu \) and \( \mu^* \) are the probability measures with respect to which \( \{p_n(x)\} \) and \( \{p_n^*(x)\} \) are orthogonal. Let \( \phi \) and \( \phi^* \) be the phase shifts in (5.3) corresponding to \( \{p_n(x)\} \) and \( \{p_n^*(x)\} \). Denote the \( n \)th convergent of the infinite continued fraction by

\[
\frac{1}{x - b_0 - \frac{a_1^2}{x - b_1 - \frac{a_2^2}{x - b_2 - \frac{a_3^2}{\ddots}}}},
\]

by \( C_n(x), n = 1, 2, \ldots \). Then for \( s = 0, 1, \ldots, k - 1 \) we have

\[
(5.4) \quad \lim_{N \to \infty} C_{Nk+s}(\cos(j\pi/k)) = \frac{1}{a_1} \sqrt{\frac{\mu'(\cos(j\pi/k))}{\mu^*(\cos(j\pi/k))}} \sin(j\pi s/k - \phi^*(j\pi/k)) \frac{\sin(j\pi s/k - \phi(j\pi/k))}{\sin(j\pi s/k + 1/k - \phi(j\pi/k))},
\]

where \( 1 \leq j < k \).

Proof. Apply (5.3) to \( p_n \) and \( a_1p_{n+1} \) then use \( C_{Nk+s}(x) = p_{Nk+s}(x)/p_{Nk+s}(x) \), with \( x = \cos(j\pi/k) \). \( \square \)
Theorem 3.1 of Andrews et al. [4] proved a version of Theorem 5.2 when \( a_n = 1, j = 1, k = 3 \), but their \( b_n \) was allowed to be complex.

**Theorem 5.3.** ([14, Theorem 42, p. 145]) Assume that

\[
\sum_{n=1}^{\infty} n \left\{ |b_n| + |a_n - 1/2| \right\} \text{ converges.}
\]

With \( \rho \) as defined in (2.4), the following limiting relation

\[
\lim_{n \to \infty} \frac{p_n(x)}{\rho(x)\rho(x-1)} = \frac{2}{\sqrt{\rho(x) - 1}} \varphi(1/\rho(x)),
\]

holds uniformly for \( |\rho(x)| > R > 1 \) where \( \varphi(1/\rho(x)) \) is a function analytic in the domain \( |\rho(x)| > 1 \).

The function \( \varphi \) is defined by

\[
\varphi(z) = \lim_{n \to \infty} \varphi_{2n}(z),
\]

and \( \varphi_{2n}(z) \) is a polynomial in \( z = e^{i\theta} \) of exact degree \( 2n \) and is defined through

\[
2i \sin \theta \rho_n(\cos \theta) = e^{i(n+1)\theta} \varphi_{2n}(e^{-i\theta}) - e^{-i(n+1)\theta} \varphi_{2n}(e^{i\theta}),
\]

see [18]. The existence of \( \varphi \) is guaranteed under (5.2), which is weaker than (5.5).

**Theorem 5.4.** Assume that (5.5) holds and let \( \varphi^*_{2n}(z) \) be the polynomials in (5.8) corresponding to \( a_1 p_{n+1}^* (\cos \theta) \) and let \( \varphi^*(z) = \lim_{n \to \infty} \varphi^*_{2n}(z) \). Then

\[
\frac{1}{x - b_0} - \frac{a_1^2}{x - b_1 - \frac{a_2^2}{x - b_2 - \frac{a_3^2}{\ddots}}} = \frac{\rho(x)}{\rho(1/\rho(x))} \frac{\varphi^*(1/\rho(x))}{\varphi^*(1/\rho(x))},
\]

for \( x \notin [-1, 1] \). Moreover the convergence is uniform for \( |\rho(x)| > R > 1 \).

Ramanujan also considered continued fractions where

\[
C_N(x, c) = \frac{1}{x - b_0 - \frac{a_1^2}{x - b_1 - \frac{a_2^2}{x - b_2 - \frac{a_3^2}{\ddots}}}} \frac{a_{N-1}^2}{x - b_{N-1} + c}
\]
Let \( \{p_n(x)\} \) and \( \{p_n^*(x)\} \) be the numerators and denominators of the continued fraction (5.5) which corresponds to \( c = 0 \). Ramanujan probably knew that

\[
C_N(x, c) = \frac{p_N^*(x) + cp_{N-1}^*(x)}{p_N(x) + cp_{N-1}(x)}.
\]

(5.11)

a fact crucial in Marcel Riesz’s treatment of the moment problem, [17, p. x, p. 47]. The polynomials \( \{p_N(x) + cp_{N-1}(x)\} \) are called quasi orthogonal polynomials, [17]. Moreover \( \{p_N^*(x) + cp_{N-1}^*(x)\} \) are their numerators. Shohat and Tamarkin use \( -\tau \) instead of \( c \) and \( \tau \) has a geometric interpretation.

The next two theorems are the versions of Theorems 5.2 and 5.4 for these continued fractions.

**Theorem 5.5.** If (5.5) holds, then

\[
\lim_{N \to \infty} C_N(x, c) = \frac{\rho(x)}{\alpha_1} \frac{\varphi^*(1/\rho(x))}{\varphi(1/\rho(x))},
\]

holds for \( x \notin [-1, 1] \).

**Proof.** Apply (5.6) to \( p_N(x) \) and \( p_N^*(x) \) and use (5.11). \( \square \)

**Theorem 5.6.** If (5.2) holds, then

\[
\lim_{N \to \infty} C_{Nk+j}(\cos(j\pi/k), c) = \frac{1}{\alpha_1} \sqrt{\frac{\mu'(\cos(j\pi/k))}{\mu^*(\cos(j\pi/k))}} \\
\times \left[ \frac{\sin(j\pi s/k - \phi^*(j\pi/k) + c\sin(j\pi(s-1)/k - \phi^*(j\pi/k))}{\sin(j\pi(s+1)/k - \phi^*(j\pi/k) + c\sin(j\pi s/k - \phi^*(j\pi/k))} \right]
\]

where \( 1 \leq j < k \).

The proof follows from Theorem 5.2 and (5.11).

**Remark 5.7.** When \( \{a_n\} \) and \( \{b_n\} \) are bounded sequences then the measure of orthogonality is unique. On the other hand, for \( x \notin \mathbb{R} \), \( C_N(x, c) \) converges to the same value for all real \( c \) if and only if the measure of orthogonality is unique, [17].

Recall the Rogers–Ramanujan continued fraction

\[
\lim_{n \to \infty} \frac{1}{1 + \frac{1}{p}} = (p^1, p^3; p^3)_{\infty} = (p^1, p^3; p^3)_{\infty}
\]

(5.12)

Denote the continued fraction on the left-hand side of (5.12) by \( RR(p) \).
Theorem 5.8. If $0 < p < 1$, then
\[
\lim_{n \to \infty} \frac{1}{p - p^3 - \cdots - p^{3n+3}} = \frac{(p^3, p^5)^\infty}{(p^2, p^3, p^5)^\infty} = RR(p)
\]

Theorem 5.9. If $0 < p < 1$, then
\[
\lim_{n \to \infty} \frac{1}{p - p^3 - \cdots - p^{3n+1}} = -RR(-p).
\]

Proof. We will only sketch the proofs of Theorems 5.8 and 5.9. As in the proofs of Theorems 4.5 and 4.6 we need to find
\[
\frac{i^{N-2}(i - 1)F(i, -q^3, -q^2) - i^{2-N}(1 - 1/i)F(-i, -q^3, -q^2)}{i^{N-1}(i - 1)F(i, q, -q^3) - i^{1-N}(1 - 1/i)F(-i, q, -q^2)} = \left\{ \begin{array}{ll}
\frac{1}{2} & iF(i, -q^3, -q^2) + iF(-i, -q^3, -q^2) \\
\frac{i}{2} & iF(i, -q^3, -q^2) - iF(-i, -q^3, -q^2)
\end{array} \right.
\]
if $N$ is odd,
\[
\frac{1}{2} & iF(i, -q^3, -q^2) - iF(-i, -q^3, -q^2)
\]
if $N$ is even.

We next evaluate these numerators and denominators as infinite products. For the denominators let
\[
H(q) = \sum_{k=0}^\infty \frac{q^{kh^2}}{(q^4, q^4)_k} = (q^2, q^4)_\infty^\infty,
\]
By formula (20) on Slater’s list, we conclude that
\[
F(i, q, -q^2) - iF(-i, q, -q^2) = H(q/i) - iH(q/i) = (1 - i)H(q),
\]
The last equality in (5.13) follows by checking the real and imaginary terms and the odd and even terms of the sum. Similarly
\[
F(i, q, -q^2) + iF(-i, q, -q^2) = H(q/i) + iH(q/i) = (1 + i)H(-q),
\]
which follows from (5.13) by replacing $i$ by $-i$ and $q$ by $-q$.
For the numerators apply (16) on Slater’s list to get
\[
H_2(q) = \sum_{k=0}^\infty \frac{q^{kh^2+2k}}{(q^4, q^4)_k} = (q^2, q^4)_\infty^\infty.
\]
Therefore we have established

\begin{equation}
\begin{aligned}
&iF(i, -q^3, -q^2) - F(-i, -q^3, -q^2) \\
&\quad = iH_2(q/i) - H_2(qi) = (i - 1)H_2(q),
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
&iF(i, -q^3, -q^2) + F(-i, -q^3, -q^2) \\
&\quad = iH_2(q/i) + H_2(qi) = (i + 1)H_2(-q),
\end{aligned}
\end{equation}

which complete the evaluation of the numerators and denominators.

\section{Al-Salam-Chihara polynomials}

The Al-Salam-Chihara polynomials \( \{Q_n(x; a, b|q)\} \) are \([5], [13]\)

\begin{equation}
\begin{aligned}
Q_n(\cos \theta) &= Q_n(\cos \theta; a, b|q) \\
&= (ab; q)_n \frac{\phi_3(q^{-n}; a, c^{-i\theta}, ac^{-i\theta}; q, q)}{a^n \phi_1(q^{-1-n}; c^{-i\theta} / a; q, q, a c^{-i\theta})},
\end{aligned}
\end{equation}

They satisfy the three term recurrence relation

\begin{equation}
\begin{aligned}
2xQ_n(x) &= Q_{n+1}(x) + (a + b)q^n Q_n(x) \\
&\quad + (1 - q^n)(1 - ab q^{n-1})Q_{n-1}(x),
\end{aligned}
\end{equation}

and the initial conditions \( Q_0(x) = 1, Q_1(x) = 2x - a - b \). On the other hand the initial conditions for \( Q_n^*(x) \) are \( Q_n^*(0) = 0, Q_1^*(x) = 2 \). The Al-Salam-Chihara polynomials satisfy the condition \((5.2)\) with

\[ a_n = \sqrt{(1 - q^n)(1 - abq^{n-1})}/2, \quad b_n = (a + b)q^n/2. \]

In this section we shall explicitly evaluate the continued fractions of \(\S 5\) for the Al-Salam-Chihara polynomials using asymptotic results.

The polynomials \( \{Q_n(x)\} \) and \( \{Q_n^*(x)\} \) have the generating functions

\begin{equation}
\sum_{n=0}^{\infty} \frac{Q_n(\cos \theta; a, b|q)}{(q; q)_n} t^n = \frac{(at, bt; q)_\infty}{(te^{-i\theta}, te^{i\theta}; q)_\infty},
\end{equation}

and

\begin{equation}
\sum_{n=0}^{\infty} \frac{Q_n^*(\cos \theta; a, b|q)}{(q; q)_n} t^n = 2t \sum_{n=0}^{\infty} \frac{(at, bt; q)_n}{(te^{-i\theta}, te^{i\theta}; q)_{n+1}} q^n,
\end{equation}

respectively, [5]. The polynomials \( \{Q_n(x)\} \) and \( \{Q_n^*(x)\} \) have the symmetry property

\begin{align*}
Q_n(-x, a, b) &= (-1)^n Q_n(x, -a, -b), \\
Q_n^*(-x, a, b) &= (-1)^{n-1} Q_n^*(x, -a, -b),
\end{align*}

\text{where}

\begin{align*}
\frac{Q_{n+1}(x)}{Q_n(x)} &= \frac{H_2(q/i)}{H_2(qi)} = (i - 1), \\
\frac{Q_{n+1}^*(x)}{Q_n^*(x)} &= \frac{H_2(q/i)}{H_2(qi)} = (i + 1),
\end{align*}

and

\begin{align*}
\frac{Q_{n+1}(q/x)}{Q_n(q/x)} &= \frac{H_2(q/i)}{H_2(qi)} = (i - 1), \\
\frac{Q_{n+1}^*(q/x)}{Q_n^*(q/x)} &= \frac{H_2(q/i)}{H_2(qi)} = (i + 1).
\end{align*}
which follow from (6.3) and (6.4). Thus we only consider the case $0 < x < 1$, 
or when $x = \cos \theta$ we may restrict ourselves to $0 < \theta < \pi/2$. With $z = e^{i\theta}$, and 
$0 < \theta < \pi$ the asymptotic formulas

$$
Q_n(\cos \theta; a, b \vert q) = \left[ \frac{a/z, b/z; q}{(q, z^{-2}, q)_{\infty}} z^n + \frac{(az, bz; q)_{\infty}}{(q, z^2, q)_{\infty}} z^{-n} \right] (1 + o(1)),
$$

(6.5)

and

$$
Q_n^*(\cos \theta; a, b \vert q) = \frac{e^{i \theta}}{i \sin \theta} 2\phi_1 \left( \begin{array}{c} \frac{ae^{-i \theta}, be^{-i \theta}}{q^{e-2i \theta}} \\ q, q \end{array} \right) + o(1)
$$

(6.6)

follow from (6.3) and (6.4), respectively.

Let $G_n(x; a, b, q)$ denote the $n$th convergent of the continued fraction

$$
G(x; a, b) := \frac{2}{2x - (a + b) - \frac{(1 - q)(1 - ab)}{2x - (a + b)q - \frac{(1 - q^2)(1 - abq)}{2x - (a + b)q - \ldots}}}
$$

(6.7)

Observe that in general, with $z = e^{i\theta}$, $0 < \theta < \pi$, then

$$
Q_n(\cos \theta; a, b \vert q) = 2z \left[ \frac{a/z, b/z; q}{(q, z^{-2}, q)_{\infty}} z^{2n+2} - \frac{(az, bz; q)_{\infty}}{(q, z^2, q)_{\infty}} \right]^{-1}
$$

$$
\times \left[ z^{2n} 2\phi_1 \left( \begin{array}{c} a/z, b/z \\ q/z^2 \end{array} \right) q, q \right]
$$

$$
- 2\phi_1 \left( \begin{array}{c} az, bz \\ qz^2 \end{array} \right) q, q \left] (1 + o(1)) \right)
$$

Apply (III.31) in Gasper and Rahman [9] with their parameters $a, b, c$ chosen as $a/z, b/z, q/z^2$ to see that

$$
2\phi_1 \left( \begin{array}{c} a/z, b/z \\ q/z^2 \end{array} \right) q, q \right) = \frac{(az^2, q)_{\infty}}{(az, bz, q)_{\infty}}
$$

$$
- \frac{(z^2, a/z, b/z; q)_{\infty}}{(z^2, az, bz; q)_{\infty}} 2\phi_1 \left( \begin{array}{c} az, bz \\ qz^2 \end{array} \right) q, q
$$
which follows from the $q$-binomial theorem. Therefore,

\[
G_n(\cos \theta; a, b, q) = -2z \left[ \frac{(a/z, b/z; q)_\infty}{(q, q^{-1}; q)_\infty} \right]^{2n+2} - \frac{(az, bz; q)_\infty}{(q, q^{z^2}; q)_\infty}^{-1} 
\times \left[ 2\phi_1 \left( \begin{array}{c} az, bz \\ q \end{array} \right| q, q^{-1} \right] \left[ 1 + z^{2n} \left( \frac{az, bz, z^2; q)_\infty}{(az, bz, z^{-2}; q)_\infty} \right) \right]^{-1} 

\]

(6.8)

Theorem 6.1. Let $k$ be an odd positive integer and $\omega_k := e^{2\pi i/k}$. With $r = 0, 1, \ldots, k - 1$, then

\[
\lim_{N \to \infty} G_{kN+r}(\cos(\pi/k); a, b, q) = 2\omega_k^{(k-1)/2} 
\times \left[ \frac{(-a\omega_k^{(k-1)/2}, -b\omega_k^{(k-1)/2}; q)_\infty}{(q, q\omega_k^{(k-1)/2}; q)_\infty} \right]^{-1} 
\times \left[ 2\phi_1 \left( \begin{array}{c} q \omega_k^{(k-1)/2} \\ q \end{array} \right| q, q^{-1} \right] \left( \frac{a\omega_k^{(k-1)/2}, -b\omega_k^{(k-1)/2}; q)_\infty}{(q, q\omega_k^{(k-1)/2}; q)_\infty} \right) 

\]

The proof follows from (6.8). A similar result can be proved for $x = \cos(j\pi/k), j = 1, 2, \ldots, k - 1$. The special case $k = 3$ is worth noting.

Theorem 6.2. With $\omega = e^{2\pi i/3}$ we have

\[
\lim_{N \to \infty} G_{3N+r-1}(1/2; a, b, q) = 2\omega(q^3; q^3)_\infty \frac{(-a, -b, q\omega^r; q)_\infty}{(-a\omega^2, -b\omega^2, q\omega^r; q)_\infty} 
\times \left[ 1 + \frac{(-a\omega, -b, q; q)_\infty}{(-a\omega^2, -b\omega^2, q^2; q)_\infty} \right] \left[ \frac{(-a\omega^2, -b\omega^2; q)_\infty}{(-a\omega, -b\omega^3; q)_\infty} \right] \left[ \frac{(-a\omega^2, -b\omega^2; q)_\infty}{(-a\omega, -b\omega^3; q)_\infty} \right] 

\]

For the $q$-ultraspherical polynomials \( \{Q_n(\cos \theta; \beta e^{i\theta}, \beta e^{-i\theta})\} \) Theorem 6.2 gives
Theorem 6.3.

\[
\lim_{N \to \infty} G_{3N+\epsilon-1}(1/2; \beta e^{i\pi/3}, \beta e^{-i\pi/3}, q) = \frac{2\omega(q^3; q^3)_{\infty} / (\beta; q)_{\infty}}{(\beta \omega^2, q\omega; q)_{\infty} \omega^x - (\beta \omega, q\omega^2; q)_{\infty}} \times \left[ 1 + \frac{2\omega(q^3; q^3)_{\infty} / (\beta; q)_{\infty}}{(\beta \omega^2, q\omega; q)_{\infty}} \right] 2\phi_1 \left( \begin{array}{c} \beta \omega, \beta \\ q, q \end{array} \right) - \frac{2\omega(q^3; q^3)_{\infty} / (\beta; q)_{\infty}}{(\beta \omega^2, q\omega; q)_{\infty}} \times \left( \beta \omega e^{i\pi/3}, \beta e^{-i\pi/3}, q \right)_{\infty} \times 2\phi_1 \left( \begin{array}{c} \beta \omega e^{i\pi/3}, \beta e^{-i\pi/3} \\ q, q \end{array} \right).
\]

Finally we consider the continued fraction when \( x \notin [-1, 1] \).

Theorem 6.4. (Askey and Ismail [5]) For \( x = (z + 1/z)/2 \) with \( |x| > 1 \) we have

\[
\lim_{n \to \infty} G_n(x; a, b, q) = G(x; a,b)
\]

(6.9)

\[
= \frac{2(q, qz^{-2}; q)_{\infty}}{z(a/z, b/z; q)_{\infty}} 2\phi_1 \left( \begin{array}{c} a/z, b/z \\ q, q \end{array} \right).
\]

In particular with \( z = q/a \)

(6.10)

\[
G((a^2 + q^2)/(2aq); a, b) = \frac{2a}{q - ab},
\]

for \( q > |a| \), follows from (6.9) and the \( q \)-binomial theorem. Similarly

\[
G((b^2 + q^2)/(2bq); a, b) = \frac{2b}{q - ab}
\]

holds when for \( q > |b| \).

Theorem 6.5. Let \( N \) be the largest integer such that \( q^{N+1} > |a| \). Then

\[
G(x_m; a, b) = \frac{2a(-a^2)^m (q^{m+2} / a^2; q)_{a^2} (q^{m+1} - ab) q^{3m+2}}{2a(q^{m+1} - ab) q^{3m+2}} \times 2\phi_2 \left( \begin{array}{c} q^{-m}, abq^{-m-1} \\ abq^{-m}, a^2 q^{-2m-1} \end{array} \right. \left. \right| \frac{q^{2m+2} + a^2}{2aq^{m+1}}, m = 1, 2, \ldots, N.
\]

where

\[
x_m = \frac{q^{2m+2} + a^2}{2aq^{m+1}}, m = 1, 2, \ldots, N.
\]

Proof. Choose \( z = q^{m+1}/a \) then apply the \( 2\phi_1 \) to \( 2\phi_2 \) transformation (III.4) in [9] to the \( 2\phi_1 \) in (6.9). The result is (6.11).

A similar theorem can be proved with \( a \) and \( b \) interchanged.
7 Remarks

In this section we briefly describe the difference between our work and [7]. One main difference is that we introduce a continuous variable $x$ into the continued fraction and formulate the convergence of the continued fraction in terms of the asymptotics of orthogonal polynomials. This forces the coefficients $b_n$ and $\lambda_n$ in (1.1) to satisfy $b_n, \lambda_n \in \mathbb{R}, n \geq 0$ and $\lambda_n > 0, n > 0$. The results in [7] allow $b_n$ and $\lambda_n$ to be complex in a way that relates $\lambda_n$ to $x$. Our results evaluate the mod $k$ convergence of the continued fractions in terms of the measures of orthogonality of the numerator and denominator polynomials, a topic not considered in [7].

To further illustrate our approach we consider the associated continuous $q$-ultraspherical polynomials where

$$b_n = 0, \quad \lambda_n = \frac{(1 - \alpha q^n)(1 - \alpha \beta q^{n-1})}{4(1 - \alpha \beta q^{n-1})}$$

Bustoz and Ismail [8] proved that the large degree asymptotics of the monic polynomials $\{P_n(\cos \theta; \beta, \alpha)\}$ is given by

$$(7.1) \quad P_n(\cos \theta; \beta, \alpha) = \frac{(\alpha; q)_{\infty}}{2n(\alpha; q)_{\infty}} \left[ 2\phi_1(\beta, \beta e^{2i\theta}; q, q; q, \alpha) \right] \times \frac{\sin[(n+1)\theta + \phi(\theta)]}{\sin \theta} [1 + o(1)],$$

where $\phi(\theta)$ is the argument of $2\phi_1(\beta, \beta e^{-2i\theta}; q, q^{-2i\theta}; q, \alpha)$. Moreover

$$(7.2) \quad P_n^*(x; \beta, \alpha) = P_{n-1}(x; \beta, \alpha + 1).$$

If $\{P_n(\cos \theta; \beta, \alpha)\}$ are orthogonal with respect to a probability measure $\mu(x; \beta, \alpha)$ then $\mu'(x; \beta, \alpha)$ is supported on $[-1, 1]$ and

$$(7.3) \quad \mu'(\cos \theta; \beta, \alpha) = \frac{2}{\pi} \left( \frac{1 - \alpha \beta}{(1 - \alpha \beta q^{n-1})} \right) \left[ 2\phi_1 \left( \beta, \beta e^{2i\theta}; q, q; q, \alpha \right) \right]^2,$$

so that $((\mu^*)'(\cos \theta; \beta, \alpha) = \mu'(\cos \theta; \beta, \alpha + 1)$.

References


